A Novel Method of Estimating Statistically Matched Wavelet: Part 1- Compactly Supported Wavelet

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ABSTRACT

Issue of finding a wavelet matched to signal has been addressed by various researchers in past. This paper presents a new method of estimating wavelet that is matched to a given signal in the statistical sense. The key idea lies in the estimation of analysis wavelet filter from a given signal and is similar to a sharpening filter used in image enhancement. The output of analysis wavelet filter branch after decimation is written in terms of filter weights and input signal samples. It is then viewed to be equivalent to difference of middle sample and its smoother estimate from the neighborhood which then needs to be minimized. To achieve this, minimum mean square error (MMSE) criterion is employed using the autocorrelation function of input signal. Since wavelet expansion acts like Karhunen-Loève type expansion for generalized $1/f^{\beta}$ processes, it is assumed that the given signal is a sample function of an n^{th} order fractional Brownian motion. Its autocorrelation function is used with MMSE criterion to estimate analysis wavelet filter. Next, a method is proposed to design 2-band FIR perfect reconstruction biorthogonal filter bank. This result in compactly supported wavelet matched statistically to given signal. Further, it is shown that compactly supported wavelet with desired support can be designed from a given signal. The theory is supported with number of simulation examples.

Keywords: FIR biorthogonal PR filter bank, Matched wavelet, $1/f^{\beta}$ processes.

1. INTRODUCTION

In the last decade, lot of work has been carried out to find wavelet matched to signal. Tewfik [2] designed wavelet matched to signal in time domain whereas Gopinath [3] found wavelet matched to signal in frequency domain for deterministic signals. Mallat and Zhang [4] proposed matching pursuits whereas best basis search for signal has been carried out by Krim et. al.[5]. Rao and Chapa [6] have proposed an algorithm to design wavelet matched to a specified signal. Similarly work has been carried out by Wu-sheng [7] and Tsatsanis [8] to design signal adapted filter banks whereas Aldroubi [9] proposed method to find matched wavelet by projecting the signal onto an existing basis.

In this paper, a new method is proposed to design a statistically matched wavelet using given signal statistics. It is well known that wavelet basis expansion acts like Karhunen-Loève type expansion for $1/f^{\beta}$ processes and are suited for the analysis of non-stationary signals. Therefore, the given signal is assumed to be a sample function of a self-similar process i.e. n^{th} order fractional Brownian motion and its autocorrelation function is used to estimate the analysis wavelet filter. Next, a method is proposed to design 2-band FIR perfect reconstruction biorthogonal filter bank. This result in compactly supported wavelet matched statistically to given signal. Further, it is shown that compactly supported wavelet with desired support can be designed from a given signal. Simulation results to validate the theory are presented for I-D signals.

Paper Outline
The paper is organized into six sections. Section 2 covers some preliminaries. A brief overview of self-similar processes and $1/f^{\beta}$ processes is provided in Section 3. Section 4 first discusses the proposed method of estimating dual wavelet filter from the given signal, assuming it to be $1/f^{\beta}$ process and then describes how to design PR biorthogonal filter bank for compactly supported wavelet based on this filter. A procedure to design wavelet with given support is also described here. Section 5 contains simulation results on synthesized $1/f^{\beta}$ signals and music/speech clips. In the end, conclusions are presented in section 6.

2. PRELIMINARIES

A wavelet system can be orthogonal or biorthogonal. Biorthogonal wavelet system relaxes the condition of orthogonality in the wavelet basis and gives more flexibility in the design process. It uses the concept of dual basis [11] where, the scaling filter f_0 and its dual h_0, wavelet filter f_1 and its dual h_1 are required to satisfy (2.1) and (2.2) for perfect reconstruction:

$$h_1(n)=(-1)^n f_0(M-n) \quad (2.1)$$
$$f_1(n)=(-1)^n h_0(M-n) \quad (2.2)$$

where, M is any odd delay.

The analysis/synthesis filter bank structure can be drawn as:

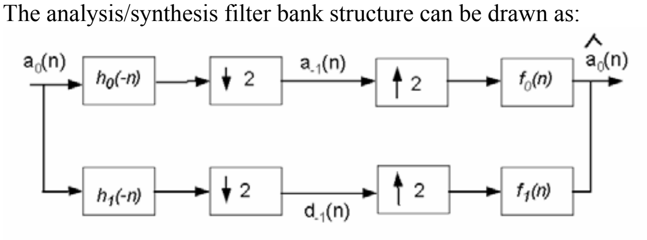


Fig.1: 2-Band Biorthogonal filter bank

The scaling function $\phi(t)$ and wavelet function $\psi(t)$ are related to $f_0(n)$ and $f_1(n)$ as:

$$\phi(t) = \sum_n f_0(n)\sqrt{2}\phi(2t-n) \qquad \forall\, n \in Z \quad (2.3)$$
$$\psi(t) = \sum_n f_1(n)\sqrt{2}\,\phi(2t-n) \qquad \forall\, n \in Z \quad (2.4)$$

Similarly, corresponding to h_0 and h_1, we get dual scaling function $\phi'(t)$ and dual wavelet function $\psi'(t)$.

3. THEORY OF SELF SIMILAR PROCESSES

A continuous time random process is called self similar if its statistical properties are scale invariant. Symbolically, it is represented as:

$$x(ct) \approx c^H x(t) \quad (3.1)$$

where, random process x(t) is self similar with self similarity index H (also called as Hurst exponent) for any scale parameter c> 0. Equality in (3.1) holds in statistical sense only.

A random process is called wide sense self-similar process if the following holds true

$$\mu_x(t) = E\{x(t)\} = c^{-H}\mu_x(ct) \quad (3.2)$$
$$r_x(t_1,t_2) = E\{x(t_1)\, x(t_2)\} = c^{-2H}\, r_x(ct_1,ct_2) \quad (3.3)$$

A continuous time stochastic process with parameter H is self similar with stationary increments (H-sssi) iff it is self similar with index H and has stationary increments. If H > 1, the increments are non-stationary, if 0 < H < 1 x(t) is H-sssi with bounded variance and if H < 0, the process is not mean square continuous.

Fractional Brownian Motion

An H-sssi Gaussian process x(t) with 0 < H < 1 is called fractional Brownian motion (FBm) and is denoted as $B_H(t)$. For value of H =1/2, the resulting process is well known Wiener process.

Though an FBm process is a non-stationary process, Flandrin[10] has shown using time-frequency representation that the averaged PSD of this process follows a power law and is directly proportional to $1/|f|^\beta$ with $\beta = 2H + 1$. Therefore, in general, these processes are also called as $1/f^\beta$ processes.

The mean value, variance and autocorrelation function $r_{BH}(t_1,t_2)$ of Gaussian $1/f^\beta$ process are given by

$$E\{B_H(t)\} = 0$$
$$Var\{B_H(t)\} = t^{2H}\sigma_H^2$$
$$r_{BH}(t_1,t_2) = 1/2\, \sigma_H^2\, (|t_1|^{2H} - |t_1-t_2|^{2H} + |t_2|^{2H}) \quad (3.4)$$

where $\sigma_H^2 = var\{B_H(1)\} = \dfrac{1}{\Gamma(2H+1)|\sin(\pi H)|}$

i.e., it is a zero mean, self similar, non-stationary random process. It is observed that normalized incremental process defined as

$$\Delta B_H(t;\varepsilon) \cong \frac{B_H(t+\varepsilon)-B_H(t)}{\varepsilon} \qquad \text{for every } \varepsilon > 0 \quad (3.5)$$

of FBm is a self-similar stationary process with parameter H′=H-1. Therefore, FBm has a generalized derivative and is termed as fractional Gaussian noise (FGn).

Corresponding to discrete data set, discrete fractional Brownian motion is defined as

$$B_H[k] = B_H[kT_s] \quad (3.6)$$

where, T_s is the sampling period. Since the process is self-similar for any value of c>0, therefore, T_s can be taken to be equal to one without loss of generality.

From (3.4) it follows that

$$E\{B_H[k]\} = 0$$
$$Var\{B_H[k]\} = k^{2H}\sigma_H^2$$
$$r_{BH}(k_1,k_2) = 1/2\, \sigma_H^2\, (|k_1|^{2H} - |k_1-k_2|^{2H} + |k_2|^{2H}) \quad (3.7)$$

Next, discrete fractional Gaussian noise can be defined as

$$X_H[k] = B_H[k] - B_H[k-1] \quad (3.8)$$

n^{th} order Fractional Brownian motion n-FBm

Fractional Brownian motion with 0 < H < 1 is called as 1-FBm and corresponding 1^{st} order incremental process is called as 1-FGn. Similarly, n-FBm process is denoted as $B_{H,n}(t)$ with n-1<H<n and corresponding n^{th} order incremental process is defined as n-FGn process. It is given as:

$$X_{H,n}(t) = \Delta_\ell^{(n)} B_{H,n}(t) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_{H,n}(t+j\ell) \quad (3.9)$$

where, ℓ is a real number and is called as lag and n is an integer.

The autocorrelation function of this n-FGn process is defined as:

$$r_{(H,n)}^{(n)}(\tau) = \frac{\sigma_H^2}{2}(-1)^n \sum_{j=-n}^{n} (-1)^j \binom{2n}{n+j} |\tau+j\ell|^{2H} \quad (3.10)$$

Again, for the discrete data set, n^{th} order discrete Fractional Brownian motion and n^{th} order discrete FGn can be considered with ℓ equal to one. The autocorrelation function of nth order discrete FGn is given by

$$r_{(H,n)}^{(n)}(k) = \frac{\sigma_H^2}{2}(-1)^n \sum_{j=-n}^{n} (-1)^j \binom{2n}{n+j} |k+j|^{2H} \quad (3.11)$$

Estimation of H Parameter

The maximum likelihood estimation method presented in [14] can be used to estimate parameter H. In [14], the method is presented for process with $0 < H < 1$ that can be easily extended to n-FBm process. If the input process is n-FBm, then its nth order incremental process will be n-FGn stationary process. Since it is stationary, ML estimation is performed using discrete n-FGn vector **X** and is denoted as $\hat{H}$:

$$\hat{H} = \max_{n-1<H<n} \left(-N\ og \frac{\mathbf{X}^T\mathbf{R}^{-1}\mathbf{X}}{N} - \log|\mathbf{R}| \right) \qquad (3.12)$$

where **R** is the autocorrelation matrix of discrete n-FGn process formed using (3.11).

4. ESTIMATION OF STATISTICALLY MATCHED WAVELET

A. Method to Estimate Dual Wavelet Filter

Consider analysis filter bank structure of two band wavelet system to which the sampled version of given signal a(t) is applied as input

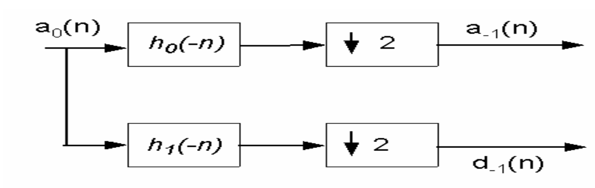


Fig.2:2-Band analysis wavelet system

$a_0(n)=a(n) \equiv$ sampled version of input signal or approximation coefficients of the signal at scale j=0.

Here, h_0 is the low pass filter and h_1 is the high pass filter such that $a_{-1}(n)$ represents the approximation coefficients at scale j=-1 and $d_{-1}(n)$ represents the finer information in wavelet subspace at scale j=-1.

Let us assume that the length of filters is N=5, then $d_{-1}(n)$ can be written in terms of filter weights as below:

$$d_{-1}(n)= h_1(0)a_0(2n)+h_1(1)a_0(2n+1) + h_1(2)a_0(2n+2)+ h_1(3)a_0(2n+3)+ h_1(4)a_0(2n+4) \qquad (4.1)$$

If the center weight $h_1(2)$ is set to unity, then, (4.1) can be rewritten as:

$$\begin{aligned} d_{-1}(n)&= a_0(2n+2) - \{-[h_1(0)a_0(2n)+ h_1(1)a_0(2n+1) + h_1(3)a_0(2n+3)+ h_1(4)a_0(2n+4)]\} \\ &= a_0(2n+2) - \hat{a}_0(2n+2) \quad = \quad e(n) \end{aligned} \qquad (4.2)$$

Discussion on equation (4.2): The equation (4.2) has been put in the above form so as to derive interesting interpretation for the same. This plays a key role in the estimation of matched wavelet. Here, $\hat{a}_0(2n+2)$ is the prediction of $a_0(2n+2)$ from the past as well as future samples. Thus, $d_{-1}(n)$ is the difference between $a_0(2n+2)$ and its average value based on the neighborhood and represents additional/finer information and mean square value of this signal should be minimized. Thus, the resulting filter h_1 should be a high pass filter. Fixing center weight equal to one will also help in the design of linear phase filter.

From (4.2), $d_{-1}(n)$ can also be represented as

$$d_{-1}(n) = e(n) = a_0(2n+j) - \mathbf{W_0}^T\mathbf{A_0} \qquad (4.3)$$

where, j = index of center weight of filter h_1

$\mathbf{W_0} = [h_1(0)\ h_1(1)\ldots h_1(j-1)\ h_1(j+1)\ldots\ldots h_1(N-1)]^T$

$\mathbf{A_0}=-[a_0(2n)\ a_0(2n+1)\ldots\ a_0(2n+j-1)\ a_0(2n+j+1)\ldots a_0(2n+N-1)]^T$

and

N = length of dual wavelet filter h_1.

$$\therefore\ E[e^2(n)] = E[a_0^2(2n+j)] - 2\,E[a_0(2n+j)\mathbf{W_0}^T\mathbf{A_0}] + E[\mathbf{W_0}^T\mathbf{A_0}\mathbf{A_0}^T\mathbf{W_0}] \qquad (4.4)$$

To minimize $E[e^2(n)]$, derivative of $E[e^2(n)]$ with respect to $\mathbf{W_0}$ is equated to zero.

$$\text{i,e,} \quad \frac{\partial E[e^2(n)]}{\partial \mathbf{W_0}} = -2\,E[a_0(2n+j)\mathbf{A_0}^T] + 2\mathbf{R_0}\mathbf{W_0} = 0$$

$$\Rightarrow \qquad E[a_0(2n+j)\mathbf{A_0}^T] = \mathbf{R_0}\mathbf{W_0} \qquad (4.5)$$

Therefore, if statistics of the input signal are known, then using (4.5) filter h_1 can be computed.

The wavelet structure is ideally suited for self-similar or say $1/f^\beta$ processes and the wavelet expansion acts like k-L type expansion for $1/f^\beta$ processes [1]. Therefore, consider input signal a(t) as self similar process with self similarity index H lying in the range $n-1 < H < n$. It can now be represented as

$$a(ct) \approx c^H a(t)$$

Or,

$$a(2t) \approx 2^H\, a(t) \qquad \text{with } c=2 \qquad (4.6)$$

The autocorrelation function of this process can be computed using (3.10). For $0 < H < 1$:

$$r_a(t_1,t_2) = 1/2\ \sigma_H^2\, (|t_1|^{2H} - |t_1-t_2|^{2H} + |t_2|^{2H})$$

$\therefore$ Using (3.3)

$$r_a(2t_1,2t_2) = 2^{2H-1}\, \sigma_H^2\, (|t_1|^{2H} - |t_1-t_2|^{2H} + |t_2|^{2H}) \qquad (4.7)$$

For discrete input process, corresponding autocorrelation function is

$$r_a(2n_1,2n_2) = 2^{2H-1}\, \sigma_H^2\, (|n_1|^{2H} - |n_1-n_2|^{2H} + |n_2|^{2H}) \qquad (4.8)$$

Using expression of (4.8), **R** for N=3 can be formed as

$$\mathbf{R}= 2^{2H-1}\, \sigma_H^2\, \mathbf{R}'$$

$$\mathbf{R}' = \begin{pmatrix} 2|2n|^{2H} & |2n|^{2H} + |2n+1|^{2H} - 1 & |2n|^{2H} + |2n+2|^{2H} - 2^{2H} \\ |2n|^{2H} + |2n+1|^{2H} - 1 & 2|2n+1|^{2H} & |2n+1|^{2H} + |2n+2|^{2H} - 1 \\ |2n|^{2H} + |2n+2|^{2H} - 2^{2H} & |2n+1|^{2H} + |2n+2|^{2H} - 1 & 2|2n+2|^{2H} \end{pmatrix} \qquad (4.9)$$

and $\mathbf{W} = [h_1(0) \quad 1 \quad h_1(2)]^T$

Thus, using (4.5) dual wavelet filter h_1 can be computed.

Initially, self-similarity index for given input signal is found by the method of [14] using (3.12). Then autocorrelation matrix **R** of $a_0(2n)$ is computed using (3.10) as in (4.8) for N=3 and $0<H<1$. Thereafter (4.5) is used to compute filter h_1 for any value of n that is sufficiently high.

B. Design of FIR Perfect Reconstruction Biorthogonal Filter Bank

The four filters h_0 , h_1, f_0 , f_1 of the analysis/synthesis filter bank structure as shown in Fig.1 are related by (2.1) and (2.2) for the condition of perfect reconstruction.

First from (2.1), the scaling filter f_0 is computed. All these filters are FIR filters. Since, the integer translates of $\phi(t)$ and $\psi(t)$ form the basis of V_0 and W_0 respectively in L^2. Similarly, $f_0(Mm-n)$ and $f_1(Mm-n)$ form the basis of ℓ^2 for integer values of m and $h_0(n-Mm)$ and $h_1(n-Mm)$ form the dual basis of ℓ^2 for integer values of m.

Therefore,

$$\sum_n h_0(n-2m_1)\, f_0(n-2m_2) = \delta(m_1-m_2) \quad \forall m_1,m_2 \in Z \qquad (4.10)$$

and
$$\sum_n h_0(n-2m_1)\, h_1(n-2m_2) = 0 \quad \forall m_1,m_2 \in Z \qquad (4.11)$$

To find h_0, (4.10) and (4.11) are required to be evaluated for only those values of m_1 and m_2 for which the vectors $f_0(n-Mm_2)$ and $h_1(n-Mm_1)$ overlap with $h_0(n)$. The filter f_1 can then be found using (2.2). Thus, all four filters can be found that form perfect reconstruction FIR biorthogonal filter bank.

C. Design of compactly supported wavelet with desired support

A compactly supported wavelet with desired support can be designed by initially choosing different order for the dual wavelet filter as well as dual scaling filter. Say, the dual wavelet filter of order 3 is chosen. And it is desired to design dual scaling filter of order 7. Then 2 extra zeros can be padded before and after the actual filter h_1 and rest of the filters can be designed using the procedure mentioned in section-4(B). Hence, a wavelet with desired support can be designed by choosing different orders for h_0 and h_1.

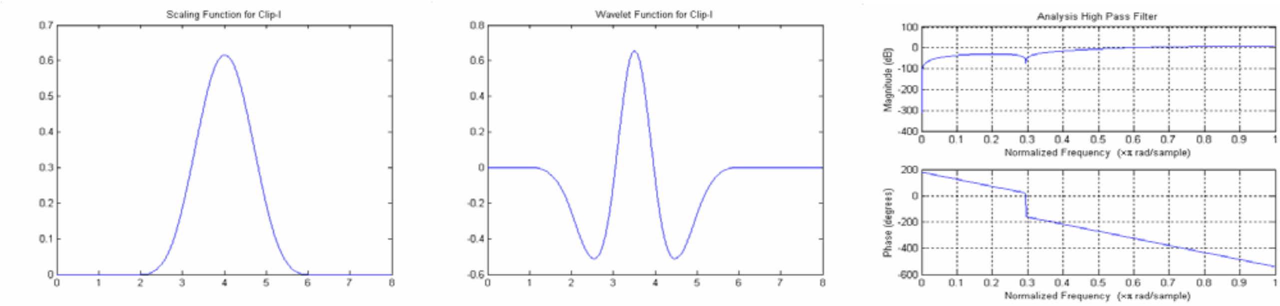
5. SIMULATION RESULTS

The proposed method is applied on two synthesized $1/f^\beta$ processes, one music, one speech and one clip with sound of horse. First dual wavelet filter is estimated using the theory proposed in section 4(A). Then, analysis/synthesis filters are designed from the signal itself for FIR PR biorthogonal filter bank based on theory presented in section 4(B).

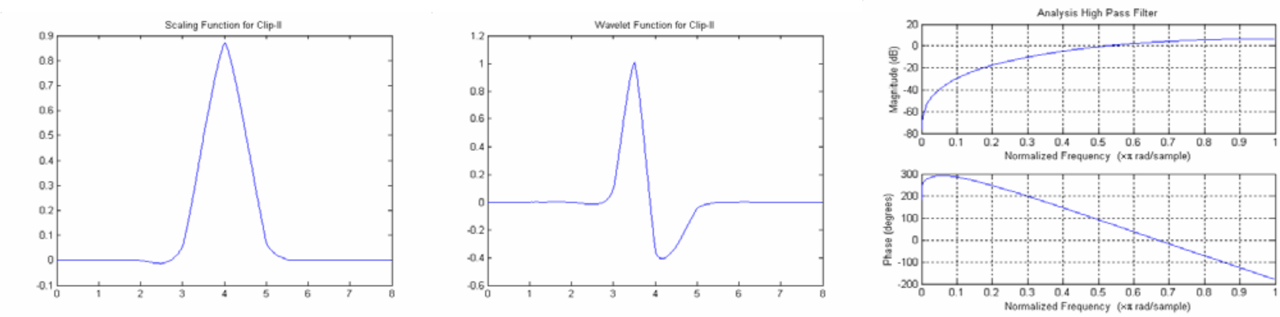
The wavelet associated with the corresponding filter bank structure is the statistically matched wavelet. The analysis and synthesis filters corresponding to statistically matched wavelet for all these five clips are tabulated in **Table-1.** Resulting wavelet and the scaling functions are plotted in **Fig.3**.

Table-1: Analysis and Synthesis filters for different clips with PR property for **statistically matched wavelet**

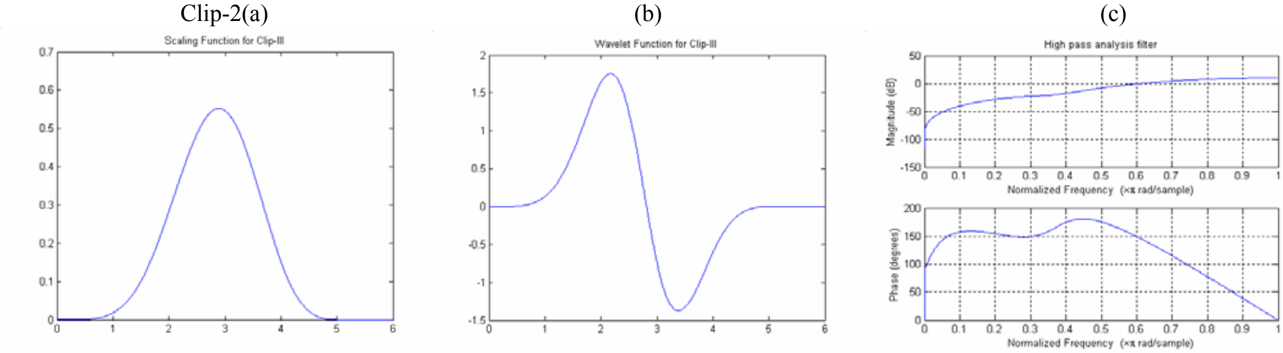
Clip No./No. of samples/Value of H	Coefficients of Analysis and Synthesis filters
1. $1/f^\beta$ clip, 4000 samples H=0.7	h_0 = [0 -0.147150 0.470914 -0.221870 -0.891308 -0.221879 0.470905 -0.147141] h_1 = [0 0.227247 -0.727242 1.0 -0.727242 0.227237 0 0] f_0 = [0 0 -0.227237 -0.727242 -1.0 -0.727242 -0.227247 0] f_1 = [-0.147141 -0.470905 -0.221879 0.891308 -0.221870 -0.470914 -0.147150 0]
2. $1/f^\beta$ clip, 3000 samples H=0.3	h_0 = [0 -0.016931 0.228188 -0.338793 -0.837994 0.027806 0.012149 0] h_1 = [0 0.040496 -0.545771 1.0 -0.552149 0.039859 0.017415 0] f_0 = [0 0.017415 -0.039859 -0.552149 -1.0 -0.545771 -0.040496 0] f_1 = [0 -0.012149 0.027806 0.837994 -0.338793 -0.228188 -0.016931 0]
3. Sound of horse, 20000 samples H=0.9508, Sampling rate: 11025Hz	h_0 =[0.412959 -1.291842 0.758511 1.690582 -1.208537 0.257451] h_1 =[0.237437 -0.742762 1.0 -0.791950 0.377747 -0.080470] f_0 = [0.080470 0.377747 0.791950 1.0 0.742762 0.237437] f_1 = [0.257451 1.208537 1.690582 -0.758511 -1.291842 -0.412959]
4. Music, 11218 samples H=1.7156, Sampling rate: 11025Hz	h_0 =[-0.194491 0.254399 -0.150729 -0.309043 -1.587747 1.209136 -0.919468 0] h_1 =[0.156738 -0.205017 -0.452637 1.0 -0.450684 -0.205475 0.156250 0] f_0 =[0 0.156250 0.205475 -0.450684 -1.0 -0.452637 0.205017 0.156738] f_1 =[0 0.919468 1.209136 1.587747 -0.309043 0.150729 0.254399 0.194491]
5. Speech, 2713 samples H=1.6568, Sampling rate: 11025Hz	h_0 = [0 -0.049773 0.175988 -0.121615 -0.286242 0.2767 05 0.727025 0.276758 -0.286330 -0.121649 0.176069 -0.049828] h_1 =[0 0 0 0.197220 -0.697327 1.0 -0.697266 0.197327 0 0 0 0] f_0 = [0 0 0 0 0.197327 0.697266 1.0 0.697327 0.197220 0 0 0] f_1 = [0.049828 0.176069 0.121649 -0.286330 -0.276758 0.727025 -0.276705 -0.286242 0.121651 0.175988 0.049773 0]



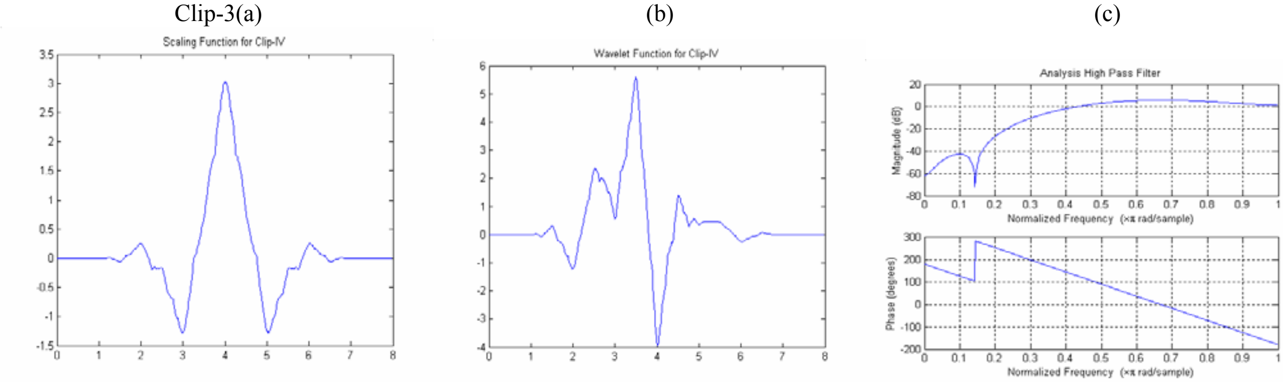
Clip-1(a) (b) (c)



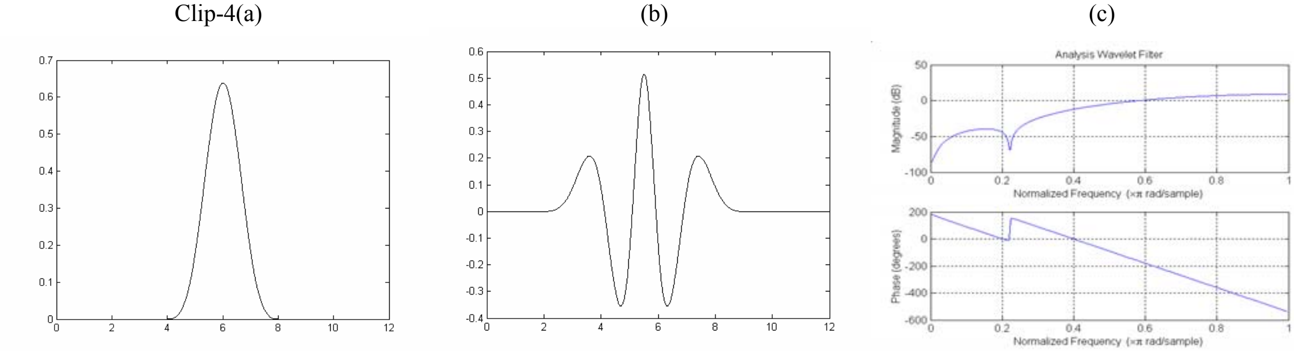
Clip-2(a) (b) (c)



Clip-3(a) (b) (c)



Clip-4(a) (b) (c)



Clip-5(a) (b) (c)

Fig 3: Statistically matched wavelets and corresponding scaling functions with dual wavelet filters for all five clips.
(a) Scaling Function (b) Wavelet Function (c) Analysis Wavelet Filter

6. CONCLUSIONS

In this paper, a new method of estimating statistically matched wavelet has been proposed. For this, first the analysis wavelet filter is extracted from signal. The idea to estimate this filter is similar to a sharpening filter used in image processing. The input signal is assumed to be n^{th} order fractional Brownian motion with zero mean. Based on this assumption, the Hurst exponent H of this self similar process is computed using the already existing methods in literature. Next, analysis wavelet filter of specific order is computed. Thereafter, a method is proposed to design 2-band perfect reconstruction FIR biorthogonal filter bank using the estimated analysis wavelet filter. The wavelet associated with this filter bank is the statistically matched compactly supported wavelet. It is shown that from a given signal, compactly supported statistically matched wavelet of desired support can be designed.

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