A Closed-Form Approximation Solution for an Inventory Model with Supply Disruptions and Non-ZIO Reorder Policy

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ABSTRACT

In supply chains, domestic and global, a producer must decide on an optimal quantity of items to order from suppliers and at what inventory level to place this order (the EOQ problem). We discuss how to modify the EOQ in the face of failures and recoveries by the supplier. This is the EOQ with disruption problem (EOQD). The supplier makes transitions between being capable and not being capable of filling an order in a Markov failure and recovery process. The producer adjusts the reorder point and the inventories to provide a margin of safety.

Numerical solutions to the EOQD problem have been developed. In addition, a closed-form approximate solution has been developed for the zero inventory option (ZIO), where the inventory level on reordering is set to be zero. This paper develops a closed-form approximate solution for the EOQD problem when the reorder point can be non-zero, obtaining for that situation an optimal reorder quantity and optimal reorder point that represents an improvement on the optimal ZIO solution. The paper also supplies numerical examples demonstrating the cost savings against the ZIO situation, as well as the accuracy of the approximation technique.

Keywords: Supply Chain Management, Reliability, Systemics, Operations Management, Optimization

1. INTRODUCTION

The fundamental EOQ model assumes that the supplier is perfectly reliable so that supply disruptions will never occur. Parlar and Perry [1996] presented in their paper an extension of this model to take supply disruptions into account. EOQ with disruptions has come to be labeled EOQD in the literature. Their model assumes that the supplier will, at any moment in time, be either in an "ON" state from which they will deliver an order instantaneously the moment the order is received, or in an "OFF" state from which they incapable of delivering anything at all. The "ON" time is exponentially distributed at a rate λ. The "OFF" time is exponentially distributed with a rate μ. A transition into the "OFF" state means a disruption of the supply chain.

When the purchaser's inventory reaches the re-order quantity *r*, then the purchaser will order *q* units from the supplier. The demand is *D* units per time period, the ordering cost is *K* per order, and the inventory holding cost is *h* per unit per time period. If the supplier is in the "ON" state, the *q* units are instantly shipped by the supplier and received by the purchaser, with the inventory rising to $q + r$ units.

If the supplier is in the "OFF" state when the purchaser's inventory reaches the re-order quantity *r*, no orders can be placed and the purchaser has to wait till the supplier returns to the "ON" state. When the supplier returns to the "ON" state, the purchaser's inventory level will have been reduced to $s' \leq r$ through sales. If the purchaser's inventory level is reduced to 0 and further demands occur, a stock-out penalty of π dollars per unit of unmet demand is incurred. When the supplier returns to the "ON" state, the purchaser orders a quantity adequate to bring inventories from *s'* up to $S' = q + r$ units. An (s', S') policy is therefore used when the supplier returns to the "ON" state following a stay in the "OFF" state. No back orders are allowed, so *s'* and *S'* are both non-negative.

All of this introduces a new trade-off not considered in the traditional EOQ model. If the purchaser decides on a small *q* and a small *r*, they risk stock-outs and lost sales. If the purchaser attempts to avoid these stock-out costs by increasing the re-order point *r* or by ordering a larger *q*, they will increase their inventories and thus their inventory carrying costs. The purchaser must trade off between both *q* and *r* to identify a cost minimizing policy (q^*, r^*).

A sample pattern of the rise and decline in inventory levels over time is shown in Figure 1.

2. BUILDING BLOCKS

The Unitary-Demand Numerical Cost Model: Parlar et al [1996] assumed that the demand $D \equiv 1$. With this assumption, they show that the expected cost of placing orders, carrying inventories and incurring outage losses is given by

$$g_0(q,r) = \frac{(K + hq^2/2 + hqr) + \beta_0 C_{10}(r)}{q + \frac{\beta_0}{\mu}}, \tag{1}$$

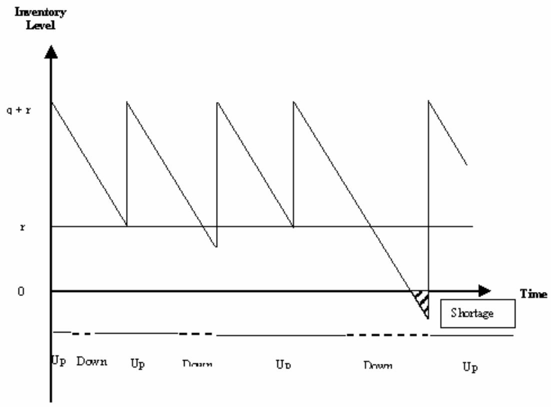

Figure 1 – Inventory model with a single supplier subject to failures and recoveries

Where β_0 is the probability that a supplier is in the "OFF" state when the purchaser's inventories reach the re-order point r, i.e.,

$$\beta_0 = \beta_0(q) = \beta(1 - e^{-(\lambda+\mu)q})$$

β is the fraction of time that the supplier is in the "OFF" state, i.e.,

$$\beta = \frac{\lambda}{\lambda + \mu}$$

$C(r)$ is the expected cost incurred from the time when the inventory reaches r and the state is "OFF" to the beginning of the next cycle when the supplier has transitioned into the "ON" state. They demonstrated that

$$C(r) = \tfrac{1}{\mu^2}(h(\mu r - 1) + e^{-\mu r}(\pi\mu + h)).$$

They then found the cost-minimizing pair (q, r), solving their model by numerical methods.

The basic EOQ model (without supply disruptions) provides solutions in closed form, explicitly giving formulas for the optimal order quantity q and the re-order quantity r, thereby showing the interrelations among q and r, the average cost, and the model parameters. However, for the EOQD model (with supply disruptions), closed form solutions that calculate the minimum of the exact cost function (1) have not been developed in the literature, including Parlar et al [1996], with numerical solutions being provided instead. However, closed-form solutions for the cost minimum as a function of the variables for q and r and the model parameters would have important and widely appreciated benefits, including the following:
Explicitly showing how values of the model's parameters and the independent variables q and r combine to generate the dependent variable of optimal cost, thereby providing a strong contextual explanation for the EOQD dynamics.
Facilitating parametric analysis and sensitivity analysis for the optimal solution.
Exploring the model's asymptotic behavior for extreme values of the various parameters.
Providing building blocks for richer and more complex models

The ZIO Closed-Form Cost Model: Snyder [2005] allowed the demand D to take on other values than 1, but restricted the reorder point r to 0 in (4), obtaining the exact cost function (2). Since r = 0 this is known as the zero-inventory ordering (ZIO) policy. Equation (2) was then approximated by replacing β_0 with β, obtaining equation (3), and closed-form solutions of (3) were then obtained.

$$g_0(q) = \frac{(K + hq^2/2D) + \beta_0 D\pi/\mu}{\frac{q}{D} + \frac{\beta}{\mu}} \qquad (2)$$

$$\beta_0 = \beta(1 - e^{-(\lambda+\mu)(q/D)})$$

$$g(q) = \frac{(K + hq^2/2D) + \beta D\pi/\mu}{\frac{q}{D} + \frac{\beta}{\mu}} \qquad (3)$$

Our Approach: In this paper, we start with the cost equation (2) and expand it to address the general case where $r \geq 0$, i.e., the non-ZIO reorder policy, obtaining a two-variable function $g_0(q, r)$. We then obtain the corresponding version of equation (3), denoting the function by $g(q, r)$, and from it develop cost-minimizing formulas in closed form. Specifically, we develop the following:
In Section 4: Theorem 1 develops the closed form formula which calculates the cost-minimizing order quantity q^* for a given $r \geq 0$.
In Section 5: Theorem 2 develops the closed form formula which calculates the cost-minimizing re-order point r^* for a given $q \geq 0$.
In Section 6: Theorem 3 develops the closed form formula that calculates the unique global cost-optimal point (q^{**}, r^{**})
In Section 7: Theorem 4 proves that the approximation $g(q, r)$ is an upper bound for the exact value $g_0(q, r)$, up to a critical value $r = \hat{r}$, above which the reverse is true.
In Section 8: Theorem 5 proves that the goodness of the relative approximation of g to g_0 is bounded above by the relative approximation between β and β_0
In Section 9: Theorem 6 proves that if q_0 is the value of q that minimizes the exact cost function $g_0(q,r)$ for a given value of r, and q^* is the value of q which minimizes the approximating function $g(q,r)$, then $q^* > q_0$ for r up to $\hat{r}$, the critical value of r identified in Section 7, above which the reverse is true.
In Section 10: We conclude with a set of application example drawn from the literature and used in Snyder [2005], as well as a replication of the numerical example in Parlar et al [1996].

3. LITERATURE REVIEW

Early papers to address the effect of disruption include Meyer, Rothkopf, and Smith [1979]. Chao [1987] discusses dynamic programming to address this problem for electrical utility companies facing possible market disruptions. Groenevelt, Pintelon, and Seidmann [1992] discuss a deterministic economic lot-sizing problem in view of machine breakdowns and corrective maintenance.

Parlar and Berkin [1991], corrected by Berk and Arreola-Risa [1994], begin a number of articles that incorporate disruptions into classical inventory models. Parlar et al [1995], mentioned previously, extends these results. Parlar and Perry [1996] extend the results in Parlar et al [1995] to the situation with two suppliers and provide direction for analysis with more than two suppliers. Gürler and Parlar [1997] allow more general failure and repair processes in the two-supplier model, presenting asymptotic results for large order quantities.

Arreola-Risa and DeCroix [1998], as well as Moinzadeh and Aggarwal [1997] consider (s,S) models with supplier disruptions. Song and Zipkin [1996] consider a situation where the availability of the supplier is partially known to the receiver. In Tomlin [2005], two supply sources exist, one cheap and unreliable and one expensive and reliable. The above papers (except for Tomlin [2005]) discuss solving the models using a numerical approach rather than in closed form, due to the complexity of the model equations.

As mentioned previously, Snyder [2005] developed an approximate closed-form model for the ZIO situation with a single supplier, where the demand is allowed to take on values other than unity. The paper develops bounds on the approximation error and addresses a power-of-two ordering policy.

4. OPTIMAL REORDER QUANTITY *Q** FOR GIVEN REORDER POINT *R*

The average cost objective function is derived from Parlar et al [1995] as

$$G_0(Q,R) = \frac{(K + HQ^2/2 + HQR) + \beta_0 \overline{C}(R)}{Q + \frac{\beta_0}{\mathrm{M}}}, \tag{4}$$

where

$$\beta_0 = \beta(1 - e^{-(\Lambda + \mathrm{M})Q})$$

$$\beta = \frac{\Lambda}{\Lambda + \mathrm{M}}$$

and

$$\overline{C}(R) = \tfrac{1}{\mathrm{M}^2}(H(\mathrm{M}R - 1) + e^{-\mathrm{M}R}(\pi\mathrm{M} + H)).$$

In Parlar et al [1995], the demand is set as $D \equiv 1$, so the values H, Λ, and M are set in terms of demand time, e.g., M = 0.2 means that an average of 0.2 recoveries occur in the period of time it takes for one unit of demand to occur, which, if the demand is $D = 100$ per year, is equivalent to 20 recoveries per year. Let h, λ, and μ be the equivalent rates for H, Λ, and M set in terms of calendar time, so that $h = HD$, $\lambda = \Lambda D$, and $\mu = \mathrm{M}D$. Then equation (4) becomes

$$G_0(Q,R) = \frac{(K + \frac{h}{D}Q^2/2 + \frac{h}{D}QR) + \beta_0 \overline{C}(R)}{Q + \frac{\beta_0}{\mu/D}} \tag{5}$$

where,

$$\overline{C}(R) = \tfrac{1}{\mu^2} D^2 (\tfrac{h}{D}(\tfrac{\mu}{D}R - 1) + e^{-\frac{\mu}{D}R}(\pi\tfrac{\mu}{D} + \tfrac{h}{D}))$$

Let $q = Q$ and $r = R/D$. Then (5) becomes

$$G_0(q, rD) = D\frac{(K + \frac{h}{D}q^2/2 + hqr) + \beta_0 \overline{C}(rD)}{\frac{q}{D} + \frac{\beta_0}{\mu}}$$

Let $g_0(q,r) = G_0(Q,rD)/D$. Then

$$g_0(q,r) = \frac{(K + hq^2/(2D) + hqr) + \beta_0 DC(r))}{\frac{q}{D} + \frac{\beta_0}{\mu}} \tag{6}$$

where

$$C(r) = \overline{C}(rD)/D = \tfrac{1}{\mu^2}(h(\mu r - 1) + e^{-\mu r}(\pi\mu + h)) \tag{7}$$

and

$$\beta_0 = \beta(1 - e^{-(\lambda+\mu)(q/D)}) \tag{8}$$

As was done in Snyder [2005] we approximate the function $g_0(q,r)$ by the function $g(q,r)$, where β_0 is replaced by β as follows

$$g(q,r) = \frac{(K + hq^2/(2D) + hqr) + \beta DC(r)}{\frac{q}{D} + \frac{\beta}{\mu}} \tag{9}$$

Note that if $r = 0$, i.e., the ZIO policy, then $C(r) = \pi/\mu$ and equations (6) and (9) reduce to the equations for $g_0(q)$ and $g(q)$ in Snyder [2005]. Also, if $D = 1$, equation (6) reduces to the equation for $g_0(q,r)$ in Parlar et al [1995]. In addition, if $\lambda = 0$, i.e., no disruptions occur, then $\beta = \beta_0 = 0$ and both equations reduce to the standard EOQ formulation.

We shall impose three reasonable assumptions on the problem parameters. First, we assume that all costs and other problem parameters including q and r are non-negative. Second, we assume that $\lambda < \mu$, i.e., the supplier "ON" states last longer than the "OFF" states.

Third, we assume that $\sqrt{2KhD} + hrD < \pi D$. If there were no disruption, the model would reduce to the classical EOQ model with a reserve inventory of r, whose optimal annual cost is $\sqrt{2KhD} + hrD$. This optimal annual cost is therefore a lower bound on the optimal cost of the system with disruptions. A feasible solution for the system with disruptions is never to place an order and maintain the reserve of r, instead stocking out on every demand; the annual cost of this strategy is πD. Since a lower bound for the optimal cost is clearly a lower bound for one of the feasible costs, this establishes the assumption.

With these parameters, formulations, and assumptions, we address the question of, for a given reorder point $r \geq 0$, what quantity $q^*(r)$ the purchaser should order so as to minimize the average cost. The closed-form result is given in Theorem 1 below:

Theorem 1: *The optimal reorder value $q^*(r)$ for a given value of $r \geq 0$, i.e., the value of q where $g(q,r)$ is a minimum, is given by*

$$q^*(r) = \frac{-\frac{\beta}{\mu}Dh + \sqrt{(\frac{\beta}{\mu}Dh)^2 + 2h(KD + \frac{\beta}{\mu^2}D^2(-h + e^{-\mu r}(\pi\mu + h)))}}{h} \tag{10}$$

for all

$$r < \tilde{r} = -\tfrac{1}{\mu}\ln\left(\frac{h - \frac{K\mu^2}{\beta}}{h + \pi\mu}\right) \tag{11}$$

If the argument of the "ln" function in (11) is negative, then (10) is true for all non-negative values of r.

Remark: If $\lambda = 0$ then $\beta = 0$ and (10) reduces to the standard EOQ result. If $r = 0$ then (10) reduces to the result shown in Snyder [2005].

Proof: To find the optimal reorder value $q^*(r)$ for a value of r we differentiate (9) with respect to q and equate to zero. Differentiating:

$$\frac{\partial g}{\partial q}=\frac{(\frac{q}{D}+\frac{\beta}{\mu})(\frac{1}{D}hq+hr)-(\frac{1}{D})(K+\frac{1}{2}hq^2/D+hqr+D\beta C(r))}{(\frac{q}{D}+\frac{\beta}{\mu})^2} \quad (12)$$

$$=\frac{\frac{1}{2}hq^2/D^2+(\frac{1}{\mu}\frac{1}{D}\beta h)q-(\frac{K}{D}+\frac{\beta}{\mu^2}(-h+e^{-\mu r}(\pi\mu+h)))}{(\frac{q}{D}+\frac{\beta}{\mu})^2}$$

Equating to zero and using the quadratic formula, the value for q^* becomes

$$q^*(r)=\frac{-\frac{\beta}{\mu}Dh\pm\sqrt{(\frac{\beta}{\mu}Dh)^2+2h(KD+\frac{\beta}{\mu^2}D^2(-h+e^{-\mu r}(\pi\mu+h)))}}{h}. \quad (13)$$

Since q^* must be nonnegative, the "±" in (10) is actually "+". In order for $q^*(r)$ to be real and positive, we must have that

$$KD+\frac{\beta}{\mu^2}D^2(-h+e^{-\mu r}(\pi\mu+h))>0, \quad (14)$$

Solving the above for the quantity $e^{-\mu r}$, condition (14) becomes

$$e^{-\mu r}>\frac{h-\frac{K\mu^2}{D\beta}}{h+\pi\mu}, \quad (15)$$

If $h-\frac{K\mu^2}{D\beta}\le 0$, then (15) holds for all r, so that $q^*(r)$ is positive for all r. If $h-\frac{K\mu^2}{D\beta}>0$,

then (15) holds for

$$r<\tilde{r}=-\frac{1}{\mu}\ln\left(\frac{h-\frac{K\mu^2}{D\beta}}{h+\pi\mu}\right)$$

The numerator of the right-hand side of (15) is clearly less than the denominator. Therefore, $\tilde{r}$ is positive and so $q^*(r)$ is positive for all values of r from zero to $\tilde{r}$.

To establish that the value of $q^*(r)$ defined in (10) is in fact a minimum, we need to establish that the second derivative of $g(q,r)$ with respect to q is positive. Differentiating (12) with respect to q, we have that

$$\frac{\partial^2 g}{\partial q^2}=\frac{(\frac{\beta}{\mu})^2\frac{h}{D}+(2K/D^2+\frac{2\beta}{D\mu^2}(-h+e^{-\mu r}(\pi\mu+h)))}{(\frac{q}{D}+\frac{\beta}{\mu})^3}. \quad (16)$$

From the discussion above, the numerator of (16) is positive for all r where Theorem 1 holds so that $q^*(r)$ is indeed a minimum.
QED

Remark The value of having a closed-form formula for $q^*(r)$ can be seen by noting the effect on $q^*(r)$ of small values of h. From inspection of (13) we have that

$$q^*(r)=-(\beta D/\mu)+\sqrt{[(\beta D/\mu)+O(1/h)]},$$

showing not just that $q^*(r)$ increases as $h\to 0$ but also showing the asymptotic properties of this increase.

5. OPTIMAL REORDER POINT *R** FOR A GIVEN ORDER QUANTITY *Q*

We next address the question of, for a given order quantity $q\ge 0$, at what inventory point r^* the purchaser should reorder so as to minimize the average cost. The closed-form result is given in Theorem 2 below:

Theorem 2: *For a given reorder quantity $q\ge 0$, the reorder inventory point $r^*(q)$ that minimizes the cost function $g(q,r)$ is given by*

$$r^*(q)=-\frac{1}{\mu}\ln\left[\frac{h(\frac{q}{D}(\frac{\mu}{\beta})+1)}{\pi\mu+h}\right] \quad (17)$$

If $q > \pi\beta D/h$, then $r^*(q) = 0$.

Proof: We differentiate g with respect to r and equate to zero, so that

$$\frac{\partial g}{\partial r}=\frac{hq+\frac{\beta D}{\mu}(h-e^{-\mu r}(\pi\mu+h))}{\frac{q}{D}+\frac{\beta}{\mu}}. \quad (18)$$

Equating the numerator to zero and solving for r we obtain

$$r^*(q)=-\frac{1}{\mu}\ln\left[\frac{h(\frac{q}{D}(\frac{\mu}{\beta})+1)}{\pi\mu+h}\right]$$

Differentiating (18) with respect to r we obtain the second derivative as

$$\frac{\partial^2 g}{\partial r^2}=\frac{\beta De^{-\mu r}(\pi\mu+h)}{q+\frac{\beta}{\mu}}, \quad (19)$$

which is positive. Therefore r^* minimizes the cost function $g(q,r)$.

If $q > \pi\beta D/h$, then $\frac{\partial g}{\partial r}>0$ for all nonnegative r, so that $r^*(q) = 0$.

QED

6. GLOBAL MINIMUM COST POINT (*Q***,*R***)

The previous sections addressed optimal values of q for specified values of r, and vice versa. The manager for the purchaser may instead wish to fully optimize the average cost by determining and using the pair of values (q^{**},r^{**}) that provides the global optimum. We now address that question, i.e., the optimal solution for the overall EOQD problem. The closed-form result is given in Theorem 3 below:

Theorem 3: *The global minimum for the EOQD function $g(q,r)$ occurs at*

$$q^{**}=\frac{D}{\mu}(1-\beta)+\sqrt{\frac{2KD}{h}+\frac{D^2}{\mu^2}(1-\beta)^2}$$

and

$$r^{**}=-\frac{1}{\mu}\ln\left[\frac{h}{\beta(\pi\mu+h)}\left(1+\sqrt{\frac{2K\mu^2}{Dh}+(1-\beta)^2}\right)\right]$$

*if r** ≥ 0. If r** < 0 then the minimum occurs at (q*(0),0), i.e., the ZIO policy (r = 0) is optimal.*

Proof: To determine the global minimum for $g(q,r)$ we need to set the two partial derivatives $\frac{\partial g}{\partial q}$ and $\frac{\partial g}{\partial r}$, given by equations (12) and (18) respectively, to zero and solve simultaneously for q and r.
The numerators for (12) and (18) are quadratic in q and linear in $e^{-\mu r}$. Let $s = e^{-\mu r}$. Then equation (10), which sets (12) to zero, can be expressed as

$$q = V + \sqrt{W + Xs}\,, \tag{20}$$

where

$$V = -\frac{\beta}{\mu}D,$$

$$W = \left(\frac{\beta}{\mu}D\right)^2 + \frac{2}{h}(KD - \frac{h\beta}{\mu^2}D^2),$$

and $X = \frac{2}{h}\frac{\beta}{\mu^2}D^2(\pi\mu + h)$,

Furthermore equation (17), which sets (18) to zero, can be expressed as

$$s = Y + Zq\,, \tag{21}$$

where

$$Y = \frac{h}{\pi\mu + h}, \text{ and}$$

$$Z = \frac{1}{D\beta/\mu}\left(\frac{h}{\pi\mu + h}\right)$$

Solving (20) for s in terms of q we obtain

$$s = \frac{(q-V)^2 - W}{X}$$

Substituting into (21) we obtain

$$q^2 - (2V + XZ)q - (W + XY - V^2) = 0$$

Using the quadratic formula, we have that

$$q^{**} = \tfrac{1}{2}\left\{(2V + XZ) \pm \sqrt{(2V + XZ)^2 + 4(W + XY - V^2)}\right\}$$

Note that $V + XY - V^2 = \frac{2K}{h}D > 0$. Therefore q^{**} is real and positive and its value is governed by the "+" part of the ± sign. After some algebra,

$$q^{**} = V + (\tfrac{1}{2}XZ) + \sqrt{2V(\tfrac{1}{2}XZ) + (\tfrac{1}{2}XZ)^2 + W + XY} \tag{22}$$

Taking ½ XZ = D/μ and $XY = 2\beta D^2/\mu^2$ and substituting into (22)

$$q^{**} = \tfrac{D}{\mu}(1-\beta) + \sqrt{\tfrac{2KD}{h} + \tfrac{D^2}{\mu^2}(1-\beta)^2} \tag{23}$$

Substituting the expressions for Y and Z into (21) we have

$$s = \frac{h}{\pi\mu + h}(1 + \tfrac{\mu}{\beta}\tfrac{q}{D})\,.$$

From (23), therefore,

$$s^{**} = \frac{h}{\beta(\pi\mu + h)}\left(1 + \sqrt{\frac{2K\mu^2}{Dh} + (1-\beta)^2}\right),$$

and

$$r^{**} = -\tfrac{1}{\mu}\ln s^{**}. \tag{24}$$

Since s^{**} is positive, r^{**} is real.

From equations (16) and (19) we have that $\frac{\partial^2 g}{\partial q^2}$ and $\frac{\partial^2 g}{\partial r^2}$ are nonnegative. In addition, we have that $\frac{\partial^2 g}{\partial q \partial r} = \frac{\partial^2 g}{\partial r \partial q} = \frac{(D\beta/\mu)e^{-\mu r}(\pi\mu + h)}{((q/D) + (\beta/\mu))^2}$, which is clearly nonnegative, so that equations (23) and (24) do yield a minimum for $g(q,r)$.

If $s^{**} > 1$, implying a negative value for r^{**}, then consider the function where $r^*(q)$ is defined in (17).

Then

$$\frac{d\tilde{g}}{dq} = \frac{\partial g}{\partial q} + \frac{\partial g}{\partial r}\frac{dr^*}{dq} \tag{25}$$

From (17)

$$\frac{dr^*}{dq} = -1/(\mu D(\tfrac{q}{D} + \tfrac{\beta}{\mu})) \tag{26}$$

Substituting (12), (18), and (26) into (25)

$$\frac{d\tilde{g}}{dq} = \frac{\tfrac{1}{2}hq^2/D^2 - \tfrac{h}{\mu D}(1-\beta)q - (K/D)}{(\tfrac{q}{D} + \tfrac{\beta}{\mu})^2} \tag{27}$$

From equation (23), q^{**} is a root of the numerator of (27), so that $\frac{d\tilde{g}}{dq} = 0$ at $q = q^{**}$. Let q^ be defined such that $r^*(q\text{^}) = 0$, i.e., $q\text{^} = \pi\beta D/h > 0$. Then since $\frac{dr^*}{dq} < 0$ from equation (26) and $r^*(q^{**}) = r^{**} < 0$, we have that $q\text{^} < q^{**}$. From inspection of (27) we have that the two roots of (27) are q^{**} and a negative value, with $\frac{d\tilde{g}}{dq} < 0$ for q in between those values, and therefore $\frac{d\tilde{g}}{dq} < 0$ for all $0 < q < q\wedge$ (note that this is the interval in which $q > 0$ and $r^*(q) > 0$), so that the minimum of $\tilde{g}$ in this interval occurs at $q = q\wedge$, or equivalently at $r = r^*(q\wedge) = 0$.

QED

Remark: Note that q^{**} is independent of the stockout cost π. The purchaser's optimal order quantity therefore does not depend on the stockout cost (though the reorder point does depend on it)

7. ABSOLUTE DEGREE OF APPROXIMATION

Having established the minimizing properties of the approximate cost function $g(q,r)$, we wish to examine its relationship to the exact cost function $g_0(q,r)$. In this section we establish that g is an upper bound for g_0 for all values of r below a specified critical value that is greater than the global optimum r^{**}. Therefore, using q and r based on $g(q,r)$ as against $g_0(q,r)$, and especially using the globally-optimizing values (q^{**},r^{**}) for $g(q,r)$, will not result in cost "surprises" due to cost underestimates.

We prove three lemmas, followed by the overall theorem (for brevity in each lemma and the theorem, the value $q^*(r)$ is denoted by q^*):

Lemma 1:

$$q^{*2} = \tfrac{2D}{h\mu}(K\mu + \tfrac{1}{\mu}(D\beta(e^{-\mu r}(\pi\mu + h) - h)) - \beta h q^*) \quad (28)$$

Proof: Equate the numerator of (12) to zero and solve for $\tfrac{1}{2}hq^*$ and then q^*.

QED

Lemma 2:

$$g(q^*,r) = h(q^* + Dr) \quad (29)$$

Proof: Substituting (28) into (9)

$$g(q^*,r) = \frac{K + \frac{1}{2D}h(\frac{2D}{K\mu}(K\mu + \frac{1}{\mu}(D\beta(e^{-\mu r}(\pi\mu + h) - h)) - \beta h q^*)) + hq^* r + \beta D C_{10}(r)}{\frac{q^*}{D} + \frac{\beta}{\mu}}$$

$$= \frac{2(K\mu D + \frac{1}{\mu}\beta D^2(e^{-\mu r}(\pi\mu + h) - h) - D\beta h q^*) + D\beta h q^* + hrD(\mu q^* + \beta D)}{q^*\mu + \beta D}$$

$$= \frac{h\mu q^{*2} + D\beta h q^* + hrD(\mu q^* + \beta D)}{q^*\mu + \beta D} \text{ (from Lemma 1)}$$

$$= \frac{hq^*(q^*\mu + D\beta) + hrD(q^*\mu + \beta D)}{q^*\mu + \beta D}$$

$$= h(q^* + Dr)$$

QED

Lemma 3:

$$\mu C(r) - hr > \sqrt{2Kh/D}, \quad (30)$$

for $r < \hat{r} = -\frac{1}{\mu}\ln\left(\frac{h + \mu\sqrt{2Kh/D}}{\pi\mu + h}\right)$ *and*

$$\mu C(r) - hr < \sqrt{2Kh/D} \quad (31)$$

for $r > \hat{r}$, *with both (30) and (31) being equations when* $r = \hat{r}$.

Remark: Note that $\hat{r} < \tilde{r}$, where $\tilde{r}$ is given in Theorem 1, so that Theorem 1 applies for all $r < \hat{r}$. Note also that since by assumption $\lambda < \mu$ so that $\beta < \tfrac{1}{2}$, it also follows that $r^{**} < \hat{r}$, where r^{**} is given Theorem 3. Therefore it is the first parts of Lemma 3 and the other lemmas below including Theorem 4 (as well as the second part of Theorem 5) that apply at the global minimum $r = r^{**}$ (and $q = q^{**}$).

Proof: From equation (7), and using some algebra, we have that

$$\mu C(r) - hr = -\tfrac{h}{\mu} + e^{-\mu r}(\pi + \tfrac{h}{\mu}) \quad (32)$$

The right-hand side of (32) is when $r = 0$, equal to π, which by assumption is greater than $\sqrt{2Kh/D}$, a decreasing function of r, when $r = \infty$, equal to $-\frac{h}{\mu}$, which is less than $\sqrt{2Kh/D}$.

There is therefore a unique value $\hat{r} > 0$ where (30) and (31) are equalities when $r = \hat{r}$, where (30) holds for all $r < \hat{r}$,and where (31) holds for all $r > \hat{r}$. Equating (32) to $\sqrt{2Kh/D}$ and solving for r we obtain $\hat{r}$ as

$$\hat{r} = -\tfrac{1}{\mu}\ln\left(\frac{h + \mu\sqrt{2Kh/D}}{\pi\mu + h}\right)$$

QED

Lemma 4:

$$\sqrt{2KhD} + hrD < g(q^*,r) < D\mu C(r) \quad (33)$$

for $r < \hat{r}$, *and*

$$\sqrt{2KhD} + hrD > g(q^*,r) > D\mu C(r) \quad (34)$$

for $r > \hat{r}$. *Both (33) and (34) are equations when* $r = \hat{r}$

Proof: For $r < \hat{r}$, $\sqrt{2Kh/D} + hr < \mu C(r)$ by Lemma 3. Substituting the definition of $C(r\}$and carrying out some algebra, we have

$$\mu\sqrt{2Kh/D} < -h + e^{-\mu r}(\pi\mu + h)$$

Multiplying both sides by 2 βh, adding $2Kh\mu^2/D + (\beta h)^2$ to both sides, completing the square on the left-hand side and rearranging terms on the right-hand side,

$$\left(\mu\sqrt{2Kh/D}+\beta h\right)^2 < (\beta h)^2 + 2h\mu[K\mu/D+\tfrac{\beta}{\mu}(-h+e^{-\mu r}(\pi\mu+h))]\cdot$$

Further algebra leads to

$$\sqrt{2KhD}+hrD < \left(-\tfrac{\beta}{\mu}Dh+\sqrt{(\tfrac{\beta}{\mu}Dh)^2+2h[KD+\tfrac{\beta}{\mu^2}(-h+e^{-\mu r}(\pi\mu+h))]}\right)+hrD$$

By equation (13), therefore,

$$\sqrt{2KhD}+hrD < hq^*(r)+hrD,$$

and by Lemma 2,

$$\sqrt{2KhD}+hrD < g(q^*,r)$$

establishing the left-hand inequality of Lemma 4.

For the right-hand inequality of Lemma 4, again by Lemma 3, $\sqrt{2hK/D} < \mu C(r)-hr$. Multiplying both sides by μ, squaring, and completing the square on the right-hand side, we have

$$\begin{aligned}&(\beta h)^2+2hK\mu^2/D+2h\beta(\mu^2C(r)-h\mu r)\\ &<(\mu^2C(r)-h\mu r)^2+2h\beta(\mu^2C(r)-h\mu r)+(\beta h)^2\end{aligned}$$

or after some algebra

$$\begin{aligned}&\tfrac{1}{\mu}\left(-\beta h+\sqrt{(\beta h)^2+2h\mu[K\mu/D+\beta(\mu C(r)-hr)]}\right)\\ &<\mu C(r)-hr)\end{aligned}$$

From equation (13) and the definition of $C(r)$, this implies that $hq^*/D + hr < \mu C(r)$. By Lemma 2, $hq^* + hrD = g(q^*,r)$, so the right-hand inequality of Lemma 4 is established.

If $r > \hat{r}$, a similar argument establishes the reverse inequalities.

If $r = \hat{r}$, a similar argument establishes the equations (note that since $\sqrt{2KhD}+h\hat{r}D = g(q^*,\hat{r}) = hq^*+h\hat{r}D$, it follows that $q^*(\hat{r})=\sqrt{2KD/h}$, i.e., the EOQ value for q).

QED

Theorem 4:

$g(q^*,r) - g_0(q^*,r) > 0$ *for* $r < \hat{r}$, *and* (35)

$g(q^*,r) - g_0(q^*,r) < 0$ *for* $r > \hat{r}$. (36)

If $r = \hat{r}$, *then* $g(q^*,r) - g_0(q^*,r) = 0$.

Proof: From equations (6) and (9), the approximation difference is given by

$$g-g_0=\frac{(\beta-\beta_0)(q^*D^2\mu^2C(r)-\mu D^2(K+\frac{1}{2D}hq^{*2}+hq^*r))}{(q^*\mu+\beta D)(q^*\mu+\beta_0D)} \quad (37)$$

If $r < \hat{r}$, we need to establish that

$$q^*D\mu C(r)-(KD+\tfrac{1}{2}hq^{*2}+hq^*rD)>0$$

This is equivalent to

$$h^2q^{*2}-2q^*hY+X<0$$

where $X = 2KDh$ and $Y = D\mu C(r)-hrD$. Completing the square, this is equivalent to

$$hq^*+\sqrt{Y^2-X}>Y$$

and

$$hq^*-\sqrt{Y^2-X}<Y$$

so that

$$\frac{Y-\sqrt{Y^2-X}}{h}<q^*<\frac{Y+\sqrt{Y^2-X}}{h}$$

Therefore, q^* must lie between the two values

$$\frac{(D\mu C(r)-hrD)\pm\sqrt{(D\mu C(r)-hrD)^2-2KhD}}{h} \quad (38)$$

From (29) and (38) the condition $g(q^*,r) > g_0(q^*,r)$ is satisfied if and only if

$$\begin{aligned}&(D\mu C(r)-hrD)-\sqrt{(D\mu C(r)-hrD)^2-2KhD}<g(q^*,r)-hrD\\ &<(D\mu C(r)-hrD)+\sqrt{(D\mu C(r)-hrD)^2-2KhD}\end{aligned}$$

The right-hand inequality is satisfied by the right-hand inequality of (33). To prove the left-hand inequality, we note that for any a, b, and c such that $a < b$ and $c \le a^2$, $b-\sqrt{b^2-c}<a-\sqrt{a^2-c}$ by the concavity of the square-root function.

Setting $a=\sqrt{2KhD}, b=D\mu C(r)-hrD, c=2KhD$, we have that $a < b$ by Lemma 4 and noting the two inequalities in (33) establishes the left-hand inequality and therefore the theorem.

If $r > \hat{r}$, then we need to establish

$$(h^2q^{*2}-2q^*hY+X)>0 \quad (39)$$

From Lemma 3 we have that $Y^2 - X < 0$, establishing (39) and proving the theorem.

If $r = \hat{r}$ then $g(q^*,r) = D\mu C(r)$ by Lemma 4, so that $q^* = \frac{1}{h}(D\mu C(r)-hDr)$ by Lemma 3. Therefore

$hq^*=Y$. Since $Y^2 = X$ by Lemma 4, we have that $h^2 q^{*2} - 2q^* hY + X = 0$, and therefore

$$q^* D\mu C(r) - (KD + \tfrac{1}{2} hq^{*2} + hq^* rD) = 0,$$

establishing that $g(q^*,r) = g_0(q^*,r)$ and proving the theorem.

QED

8. RELATIVE DEGREE OF APPROXIMATION

We wish to investigate the relative degree of approximation of g to g_0, i.e.,

$$\frac{g(q^*,r) - g_0(q^*,r)}{g_0(q^*,r)}$$

in particular the degree to which the quantity $(\beta - \beta_0)/\beta_0$ is an upper bound to this degree of approximation. Since, for many values of λ, μ, and q^*, the values β and β_0 are very close to each other, such an upper bound will be a powerful statement on the value of the approximation.

Theorem 5: *Let* $g_E(q,r) = \frac{KD}{q} + \frac{hq}{2} + hr$ *be the classical EOQ cost function with a nonnegative reorder point. Then:*

a) For all $q>0$ *such that* $g_E(q,r) < D\mu C(r)$,

$$\frac{g(q,r) - g_0(q,r)}{g_0(q,r)} < \frac{\beta - \beta_0}{\beta_0}\left(1 - \frac{g_E(q,r)}{D\mu C(r)}\right) < \frac{\beta - \beta_0}{\beta_0} = \frac{e^{-(\lambda+\mu)(q/D)}}{1 - e^{-(\lambda+\mu)(q/D)}}$$

b) If $r < \hat{r}$ *then* $g_E(q^*,r) < D\mu C(r)$, *so that*

$$\frac{g(q^*,r) - g_0(q^*,r)}{g_0(q^*,r)} < \frac{\beta - \beta_0}{\beta_0}\left(1 - \frac{g_E(q^*,r)}{D\mu C(r)}\right) < \frac{\beta - \beta_0}{\beta_0} = \frac{e^{-(\lambda+\mu)(q^*/D)}}{1 - e^{-(\lambda+\mu)(q^*/D)}}$$

c) If $r > \hat{r}$ *then*

$$\frac{g_0(q^*,r) - g(q^*,r)}{g_0(q^*,r)} < \frac{\beta - \beta_0}{\beta}\left(1 - \frac{D\mu C(r)}{g_E(q^*,r)}\right) < \frac{\beta - \beta_0}{\beta} = e^{-(\lambda+\mu)(q^*/D)}$$

If $r = \hat{r}$ *then* $(g(q,r) - g_0(q,r))/g_0(q,r) = 0$.

Proof:

a) From (37)

$$g(q,r) - g_0(q,r) = \frac{(\beta - \beta_0)D\mu(qD\mu C(r) - (KD + \frac{1}{2}hq^2 + hqDr))}{(q\mu + \beta D)(q\mu + \beta_0 D)}$$

Therefore,

$$\frac{g(q,r) - g_0(q,r)}{g_0(q,r)} = \frac{(\beta - \beta_0)(D\mu C(r) - (KD/q + \frac{1}{2}hq + hDr))}{(q\mu/D + \beta)(KD/q + \frac{1}{2}hq + hDr)} \frac{KD/q + \frac{1}{2}hq + hDr}{KD/q + \frac{1}{2}hq + hDr + D^2\beta_0 C(r)/q}$$

$$= \frac{(\beta - \beta_0)(D\mu C(r) - g_E(q,r))}{(q\mu/D + \beta)g_E(q,r)} \frac{g_E(q,r)}{g_E(q,r) + D^2\beta_0 C(r)/q}$$

$$= \frac{(\beta - \beta_0)}{(q\mu/D + \beta)(1 + D^2\beta_0 C(r)/qg_E(q,r))}\left(\frac{D\mu C(r)}{g_E(q,r)} - 1\right)$$

$$< \frac{(\beta - \beta_0)}{(q\mu/D)(D^2\beta_0 C(r)/qg_E(q,r))}\left(\frac{D\mu C(r)}{g_E(q,r)} - 1\right)$$

$$= \frac{(\beta - \beta_0)}{\beta_0}\left(1 - \frac{g_E(q,r)}{D\mu C(r)}\right) < \frac{(\beta - \beta_0)}{\beta_0} = \frac{\exp(-(\lambda+\mu)(q/D))}{1 - \exp(-(\lambda+\mu)(q/D))}$$

b) From the quadratic formula, (38) is equivalent to

$$\frac{KD}{q^*} + \frac{hq^*}{2} + hrD = g_E(q^*,r) < D\mu C(r)$$

The result therefore follows from part (a).

c) From part (a) we have that

$$\frac{g_0(q,r) - g(q,r)}{g_0(q,r)} = \frac{(\beta - \beta_0)}{(q\mu/D + \beta)(1 + D^2\beta_0 C(r)/qg_E(q,r))}\left(1 - \frac{D\mu C(r)}{g_E(q,r)}\right)$$

Therefore,

$$\frac{g_0(q,r) - g(q,r)}{g_0(q,r)} < \frac{(\beta - \beta_0)}{\beta}\left(1 - \frac{D\mu C(r)}{g_E(q,r)}\right) < \frac{(\beta - \beta_0)}{\beta} = e^{-(\lambda+\mu)(q^*/D)}$$

d) This part follows directly from Theorem 4

QED

9. UPPER (OR LOWER) BOUND ON THE EXACT REORDER LEVEL

In addition to the approximating cost function g being an upper bound for the exact cost function g_0, we can also establish for what values of the reorder point r the optimal order quantity q^* for the approximating cost function is an upper (or lower) bound for the optimal order quantity q_0 for the exact cost function.

Theorem 6: *If* $r < \hat{r}$, *then* $q^* > q_0$, *where* q_0 *minimizes* $g_0(q,r)$. *If* $r > \hat{r}$, *then* $q^* < q_0$, *and if* $r = \hat{r}$ *then* $q^* = q_0$.

Proof: Since from equation (8) $\dfrac{d\beta_0}{dq} = \dfrac{\lambda}{D} e^{-(\lambda+\mu)(q/D)}$, the first derivative of g_0 with respect to q is given by

$$\frac{\partial g_0}{\partial q} = \frac{1}{\frac{q}{D}+\frac{\beta_0}{\mu}} \left(\begin{array}{l} (\frac{q}{D}+\frac{\beta_0}{\mu})(\dfrac{h}{D}q + hr + \lambda \exp(-(\lambda+\mu)(q/D))C(r)) \\ -(K + \dfrac{h}{2D}q^2 + hrq + \beta_0 DC(r))(\dfrac{1}{D} + \dfrac{\lambda}{\mu D}\exp(-(\lambda+\mu)(q/D))) \end{array} \right)$$

The numerator can be rewritten as

$$\begin{array}{l} \frac{h}{2D^2}q^2 + \frac{\beta}{D\mu}hq + \frac{\beta}{\mu}hr - \frac{K}{D} - \beta C(r) \\ + e^{-(\lambda+\mu)(q/D)}\left(-\frac{\lambda h}{2\mu D^2}q^2 + \frac{\lambda}{D}C(r)q - \frac{\beta}{D\mu}hq - \frac{\lambda hr}{D\mu}q - \frac{\lambda K}{D\mu} - \frac{\beta}{\mu}hr + \beta C(r)\right) \end{array}$$

and further rewritten as

$$\begin{array}{l} \frac{h}{2D^2}q^2 + \frac{\beta}{D\mu}hq + \frac{\beta}{\mu}hr - \frac{K}{D} - \beta C(r) \\ + e^{-(\lambda+\mu)(q/D)}\left(-\frac{h}{2D^2}q^2 - \frac{\beta}{D\mu}hq - \frac{\beta}{\mu}hr + \frac{K}{D} + \beta C(r)\right) \\ + e^{-(\lambda+\mu)(q/D)}\left(\frac{h}{2D^2}(1-\frac{\lambda}{\mu})q^2 + \frac{\lambda}{D}C(r)q - \frac{\lambda hr}{D\mu}q - \frac{K}{D}(1+\frac{\lambda}{\mu})\right) \end{array} \qquad \text{(40a,b,c)}$$

The expression in (40a) is the numerator for $\dfrac{\partial g}{\partial q}$ and the parenthetical term in (40b) is its negative, so that when $q = q^*$ both of these terms are zero. Using (29), the second factor of (40c) can be rewritten as

$$\left(\tfrac{1}{\mu D}(1-\tfrac{\lambda}{\mu})(K\mu + (\mu C(r) - hr)\beta - \beta hq) + \tfrac{\lambda}{D}C(r)q - \tfrac{\lambda hr}{D\mu}q - \tfrac{K}{D}(1+\tfrac{\lambda}{\mu})\right)$$

$$= \left(\tfrac{\beta}{\mu D}(1-\tfrac{\lambda}{\mu})(\mu C(r) - hr - hq) + \tfrac{\lambda}{\mu}(-2\tfrac{K}{D} + (\tfrac{\mu}{D}C(r) - \tfrac{hr}{D})q)\right) \quad (41)$$

We assume that $\lambda < \mu$, so that $(1-\frac{\lambda}{\mu}) > 0$. If $r < \hat{r}$, then by Lemma 3, $hDr + hq^* = g(q,r^*)$, and by Lemma 1, $g(q,r^*) < D\mu C(r)$, so that $D\mu C(r) - hDr - hq^* > 0$. Similarly, $D\mu C(r) - hDr > hq^*$, so that $q^*(D\mu C(r) - hDr) > hq^{*2}$. Also by Lemma 2 and Lemma 3, $\sqrt{2KhD} + hDr < hq + hDr$, so that $hq^2 > 2KD$. Hence the expression in (41) and therefore the second term of (40c) are both positive. Therefore $\partial g_0 / \partial q$ is positive when $q = q^*$. Since g_0 is unimodal (by a similar argument as Proposition 2b in Berk et al [1994], it must attain its minimum to the left of q^*, therefore $q^* > q_0$.

If $r > \hat{r}$, then a similar argument shows that $\partial g_0 / \partial q$ is negative when $q = q^*$, so that g_0 must attain its minimum to the right of q^*, so that $q^* > q_0$.

If $r = \hat{r}$, then a similar argument shows that $\partial g_0 / \partial q$ is zero when $q = q^*$, so that $q^* = q_0$.

QED

10. COMPUTATIONAL RESULTS

We tested our formulas using the 10 sets of parameters *h, K, π,* and *D* adopted in Snyder [2005], which are in turn derived from sample problems for the "(*Q,R*)" model found in production and inventory textbooks. These parameter sets are displayed in Table 1. For each pair of failure rate λ and recovery rate μ, we therefore have 10 sets of parameters. We selected 16 failure and recovery rate pairs, using 4 values of λ (i.e., 0.5, 1, 2, and 5) and 4 values of μ (i.e., 2λ, 4λ, 10λ, and 20λ) for each value of λ. This results in 160 total application examples.

We also tested our formulas on the numerical example given in Parlar et al [1995], shown in the last rows of each of the tables that follow.

Table 1 – Problem parameters

Set	*h*	*K*	*π*	*D*
1	0.8	30	12.96	540
2	15.0	10	40.00	14
3	6.5	175	12.50	2000
4	2.0	50	25.00	200
5	45.0	4500	440.49	2319
6	5.0	300	50.00	3000
7	0.0132	20	0.34	1000
8	5.0	28	80.00	520
9	0.005	12	0.12	3120
10	3.6	12000	65.73	8000
Parlar & Perry	5	10	260	1

For each application example, we computed

- The optimal ZIO (i.e., $r = 0$) order quantity $q^*(0)$, using the closed-form approximating function *g,* and the value of *g* and the exact function g_0 for $q = q^*(0)$ and $r = 0$
- The optimal reorder inventory point $r^*(q)$ for the order quantity $q = q^*(0)$, and the value of *g* and g_0 for
- $q = q^*(0)$ and $r = r^*(q^*(0))$
- The globally optimal point (q^{**},r^{**}) using the closed-form approximating function g, and the value of *g* and g_0 at (q^{**},r^{**})
- The globally optimal point (q_0,r_0) using the exact function g_0 (obtained via numerical methods)

A Sample Cost Analysis: Figure 2 shows a sample plot of the approximate and exact cost functions *g* and g_0 for one of the sample sets ($\lambda = 2$, $\mu = 20$, and the sixth set of parameters in Table 1). The upper set of curves shows the approximate and exact cost functions *g* and g_0 using the ZIO policy (i.e., $R = 0$ for each value of *Q*), while the lower shows the two functions using the optimal reorder policy (i.e., $R = R^*(Q)$ for each value of *Q*).

- Text box 0 shows the EOQ policy in the absence of supplier failures.
- Text box 1 shows the EOQ policy in the presence of supplier failures
- Text box 2 shows the ZIO policy ($Q = Q^*$ and *R* is zero)

- Text box 3 shows the ZIO order quantity with the optimal reorder point ($Q = Q^*$ and $R = R^*(Q^*)$).
- Text box 4 shows the overall optimum (Q^{**},R^{**}) using the function g.
- Text box 5 shows the overall optimum (Q_0,R_0) using the function g_0 (solved numerically).

We obtain the following significant conclusions:

- The cost of unreliability is significant. With a perfectly reliable supplier the average costs would have been \$3000. Because of the unreliability, the costs become considerably higher.

- Ignoring the failure situation by ordering only the EOQ and not maintaining a positive reorder inventory results in significantly higher costs (\$6267).

- By increasing the order quantity from the EOQ level of 600 to the ZIO level of 1072, the manager of the purchaser garners definite savings (the costs become \$5359). This is the improvement shown in Snyder [2005].

- While keeping the order quantity at the ZIO level of 1072, significant savings beyond the ones in Snyder [2005] can be generated by maintaining a positive reorder inventory of 139 units. This is the result shown in Theorem 2 (the costs become \$4910)

- By using the closed-form optimal order quantity and reorder point in Theorem 3, the manager of the purchaser can gain still further savings beyond any ZIO-level order quantity policy. With the optimal order quantity of 752 and reorder point of 191, the costs become \$4715.

- Using numerical methods to obtain the exact global optimum generates only a negligible amount of further savings (the costs become \$4712 instead of \$4715). Generally, the approximate and exact cost functions are very close to each other.

Accuracy of the Approximation: Table 2 addresses the accuracy of the approximate cost function g against the exact cost function g_0 at the global minimum for g (i.e., $g(q^{**},r^{**})$ vs. $g_0(q^{**},r^{**})$). The approximation is at its most accurate when the supplier is at their most unreliable and/or can recover quickly. The average relative accuracy is under 1%, with the best average accuracy virtually exact and the worst 3.1%. The approximate function is always an upper bound for the exact function.

Figure 2 -- Average Cost Function

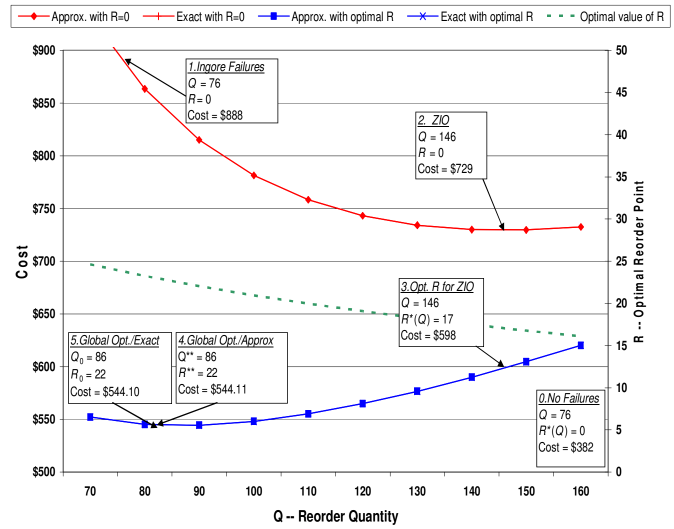

Table 2 – Relative Accuracy of Cost Function (Approximating cost function g vs. exact cost function g_0 at g's global minimum (q,r**))**

λ	μ	Best	Average	Worst
0.5	1	0.4%	3.7%	11.6%
0.5	2	0.1%	3.3%	11.3%
0.5	5	<0.1%	1.7%	7.0%
0.5	10	<0.1%	0.5%	2.1%
1	2	<0.1%	1.8%	5.5%
1	4	<0.1%	1.4%	4.9%
1	10	<0.1%	0.4%	1.9%
1	20	<0.1%	0.1%	0.3%
2	4	<0.1%	0.8%	2.6%
2	8	<0.1%	0.4%	1.6%
2	20	<0.1%	<0.1%	0.2%
2	40	<0.1%	<0.1%	<0.1%
5	10	<0.1%	0.1%	0.5%
5	20	<0.1%	<0.1%	0.1%
5	50	<0.1%	<0.1%	<0.1%
5	100	<0.1%	<0.1%	<0.1%
Average		0.1%	0.9%	3.1%
Parlar & Perry				
0.25	2.5	Relative Accuracy<0.1% (0.02%)		

Cost Savings of Optimal Reorder Point vs. ZIO Option: Table 3 shows the improvement, in terms of percent decline in cost, when the reorder inventory point r is allowed to move to its optimum r^* instead of the ZIO assumption for the order quantity $q^*(0)$ value under consideration, i.e., replacing $g(q^*(0),0)$ by $g(q^*(0),r^*(q^*(0))$. The results show that the situation does improve when one keeps a reserve inventory on

reordering. The average improvement is 5.2%, an appreciable cost savings.
As might be expected, the most significant improvement takes place in the most adverse situation, when the supplier is the most unreliable and least recoverable. For the strongest such situation, i.e., $\lambda = 5$ and $\mu = 10$, i.e., the supplier is "OFF" one-third of the time, the average cost savings is 14.2%, e.g., a savings of $1420 in a supply chain with $10000 of costs.

Table 3 – Cost Savings (optimal reorder point vs. zero reorder point, for a ZIO-based optimal reorder quantity)

λ	μ	Least	Average	Most
0.5	1	0%	4.2%	7.7%
0.5	2	0%	3.8%	6.7%
0.5	5	0%	1.9%	6.8%
0.5	10	0%	0.8%	5.6%
1	2	0%	7.8%	11.8%
1	4	0%	6.5%	12.5%
1	10	0%	3.3%	11.9%
1	20	0%	1.3%	8.9%
2	4	1.0%	11.2%	17.7%
2	8	0.3%	8.5%	18.3%
2	20	0%	4.0%	16.1%
2	40	0%	1.5%	10.1%
5	10	4.0%	14.2%	24.8%
5	20	0.9%	9.5%	24.3%
5	50	0%	3.7%	18.0%
5	100	0%	1.1%	8.3%
Average		0.4%	5.2%	13.1%
Parlar & Perry				
0.25	2.5	Cost Savings = 2.5%		

Cost Savings of Globally Optimal Reordering vs. ZIO Option: Table 4 shows the improvement, in terms of percent decline in cost, when the globally-optimal pair of reorder quantity and reorder inventory is used in place of the ZIO policy (i.e., $g(q^{**},r^{**})$ vs. $g(q^{*}(0),0)$). The results show that adjusting the reorder quantity as well as the reorder inventory point adds to the cost savings. The average cost savings is 8.5%, which is more than a 50% improvement from that obtained by simply adjusting the reorder inventory point (i.e., Table 3).
Again, as might be expected, the most significant improvement takes place in the most adverse situation, when the supplier is the most unreliable and least recoverable. For the situation where $\lambda = 5$ and $\mu = 10$, the average cost savings is 22.7%, e.g., a savings of $2270 in a supply chain with $10000 of costs. Since the overall supply chain costs in this adverse situation are likely to be higher than normal, the higher cost savings occur in precisely the situation where these savings probably are urgently needed.

Table 4 – Cost Savings (globally optimal reorder point vs. zero reorder point)

λ	μ	Least	Average	Most
0.5	1	0%	8.5%	14.3%
0.5	2	0%	6.8%	13.9%
0.5	5	0%	3.2%	13.0%
0.5	10	0%	1.3%	9.6%
1	2	0%	14.6%	24.6%
1	4	0%	11.2%	25.1%
1	10	0%	5.2%	21.5%
1	20	0%	1.9%	13.7%
2	4	2.1%	19.7%	35.9%
2	8	0.5%	13.8%	35.0%
2	20	0%	5.7%	26.4%
2	40	0%	1.9%	13.8%
5	10	5.7%	22.7%	47.5%
5	20	1.1%	14.0%	42.5%
5	50	0%	4.7%	25.4%
5	100	0%	1.3%	9.9%
Average		0.6%	8.5%	23.3%
Parlar & Perry				
0.25	2.5	Cost Savings = 3.2%		

11. SUMMARY AND CONCLUSIONS

In this paper, we have extended the state of the art in modeling supply chain performance in the face of reliability disruptions, i.e., the EOQD situation, to provide closed-form closely-approximate solutions that go beyond the ZIO-policy or numerical approaches currently in the literature. The results allow managers to carry out tradeoffs between the order quantity and the reorder point, as well as to determine the best order-quantity/reorder-point pair in terms of minimizing average total cost. With the results in closed form, managers and analysts can carry out parametric, asymptotic, and extreme-value analyses of the EOQD situation, as well as incorporate the cost models into broader supply-chain and overall business analysis evaluations.

Specifically, we have developed closed-form equations for the optimal order quantity given an inventory reorder point, the optimal inventory reorder point given an order quantity, and the globally optimal order-quantity/inventory-reorder point. We have shown that the closed form results provide a close upper bound to the exact cost for a range of reorder-point values from zero on up. We have established these results on application examples appearing in the EOQD literature.

Possible further research includes extending the results in this paper to address multiple supplier sources, multiple-component supply chain, and multi-echelon supply chains. Additional research includes incorporating the results into broader models, exploiting their closed-form nature.

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