Geometry of Nash Equilibrium in Quantum Hawk-Dove Games

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ABSTRACT

A game is said to be “quantized" when the expected payoff to the player(s) is computed via the higher order randomization notion of quantum superposition followed by measurement versus the randomization notion of probability distribution. A major motivation for quantizing a game is the potential manifestation of Nash equilibria that are superior to those already available in the game. Quantum superpositions are elements of a (projective) Hilbert space which, among other things, is an inner product space. The inner product of the Hilbert space of quantum superpositions is used here to give a geometric characterization of Nash equilibrium in quantized versions of Hawk-Dove games, a class of games to which the well known game Prisoners' Dilemma belongs.

1. INTRODUCTION

Multi-player game theory can informally be described as the mathematical study of conflict and cooperation between various interacting individuals. Call the interaction a *game*, the individuals involved *players*, and the ability of a player to interact with the other players his *pure strategies*. Suppose also that each player has stakes in the game called the *payoffs* and that each player is *rational*, that is, she will seek to maximize her payoffs in a manner consistent with some preference relation over the payoffs. A *play* of the game now entails the choice of a pure strategy by each player the result of which is a tuple of pure strategies called a *pure strategy profile*. Payoff to the players is determined by the particular pure strategy profile employed. In this context, each player will choose a pure strategy that is a <u>best reply</u> to his opponent’s choice of pure strategy, thus maximizing his payoff. If every player succeeds in finding such a strategy, then the resulting pure strategy profile is called a *Nash equilibrium*. Identification of Nash equilibria is a fundamental goal of multi-player game theory.

In a given game however, Nash equilibria are not necessarily optimal. Worse, they may not even exist. In such cases, Von Neumann calls upon the players to enlarge their strategy sets to include *mixed strategies*, that is, randomization between their pure strategies via probability distributions. The use of mixed strategies results in probability distributions over the pure strategy profiles and the payoffs are now computed as expected payoffs. When the strategy sets of players are finite, the merit of using mixed strategies arises from Nash's famous theorem which state's that equilibrium always exists in terms of mixed strategies. Moreover, it is often the case that such equilibrium is optimal or close to optimal.

Enlarging the set of strategies available to the players in a game is not merely a time honored heuristic. It is in fact a mathematically sound procedure in the following sense. A game can be viewed formally in terms of its *payoff function* which takes a pure strategy profile to a *payoff profile,* a tuple of real numbers that assigns to each player the payoff corresponding to the player's particular choice of strategy in the strategy profile. As such, the use of mixed strategies in a game amounts to extending the domain of the payoff function to include probability distributions, resulting in what is often called a ``mixed'' game. This extension is ``proper'', that is, the extended game can always be restricted to the pure strategies to recover the original game. Proper extensions allow for a meaningful comparison between the results generated by the extended game and the original one. From now on, no distinction will be made between a game and its payoff function.

Other extensions are possible. One extension, proposed by Meyer [4] about a decade ago, allows players to utilize *pure quantum strategies*, that is, sets of qudits together with quantum operations on them. The use of pure quantum strategies results in a higher order randomization between the pure strategy profiles via *quantum superpositions* which are complex projective linear combinations followed by *measurement*, that is, orthogonal projection. Expected payoff is now computed via the probability distribution over pure strategy profiles that results from measurement. Such an extension of a game is known as *quantization* of the game, and the resulting game itself is called a ``quantum'' game. Since the fundamental idea behind game quantization is that of forming quantum superposition of pure strategy profiles, Bleiler [2] has recently proposed that pure quantum strategies be *any* non-empty set. The area of research that studies quantum games is known as quantum game theory and a major consideration in the subject is the appearance of ``new'' optimal or close to optimal Nash equilibria in terms of quantum strategy profiles.

Unlike extensions to mixed games however, quantizations are not automatically proper. This somewhat subtle fact has in the past led to questions about the relevance of the Nash equilibria that manifest in improperly quantized games to the corresponding classical game. Such issues were recently resolved by Bleiler in [2] via a mathematically formal approach to quantization of games in terms of domain extension. In the language of the Bleiler formalism for ``quantum mixing'', quantizations from which the original game and the mixed game can be recovered upon restriction of the domain are, respectively, *proper* and *complete* quantizations. Note that a complete quantization is automatically proper. Both proper and complete quantizations make it game theoretically meaningful to speak of ``new'' Nash equilibria in quantized games.

	II	
I	Dove	Hawk
Dove	$(\frac{1}{2}V, \frac{1}{2}V)$	$(0, V)$
Hawk	$(V, 0)$	$(\frac{1}{2}V - C, \frac{1}{2}V - C)$

Figure 1. A representative from the family of Hawk-Dove games.

A complete quantization of the popular Hawk-Dove game Prisoner's Dilemma is proposed by Eisert, Wilkens, and Lewenstien (EWL) in [3]. These authors show that a new optimal Nash equilibrium appears in the game for quantum strategy profiles consisting of a certain sub-class of quantum strategies. However, when quantum strategy profiles consisting of the most general class of quantum strategies are employed, the *only* Nash equilibrium that manifests is the sub-optimal one in terms of the players' original pure strategies. However, a further extension of the game to include *mixed quantum strategies*, that is, probability distributions over the pure quantum strategies of the players, results in a Nash equilibrium in which each player gets a payoff close, but not equal to, the optimal payoff in the game.

Pure quantum strategy Nash equilibria in quantum Hawk-Dove games are typically computed by analyzing the probability distributions over the outcomes that result from the measurement of a corresponding quantum superposition, as in [3] for instance. In this article, the notion of Nash equilibrium in terms of pure quantum strategies of players in Hawk-Dove games is characterized in terms of the geometry of the state space of quantum superpositions. Note that mixed quantum strategies are not quantum superposition, but rather probability distributions over pure quantum strategies. As such, mixed quantum strategies play no further role in this article.

2. HAWK-DOVE GAMES

Hawk-Dove games typically refer to a class of two player games in which each player has access to two strategies. As Binmore [1] describes it, such games arise when two members of the same bird species compete for territory the value of which is $V > 0$. Each bird can adopt a Hawkish or a Dovish strategy. If both birds behave like Doves, they split the territory with a value of $\frac{1}{2}V$ each. If one behaves Dovish and the other Hawkish, then the latter gets the entire territory. If both birds behave Hawkish, a fight, carrying a cost $C > 0$, ensues and the resulting value of the territory that each bird gets reduces to $\frac{1}{2}V - C$. The strategic form of an arbitrary Hawk-Dove game is given in Figure 1 in which the rows form the strategies of, say, player 1, and the columns form the strategies of player II. The first entry in each payoff vector in Figure 1 is the payoff to player I while the second entry is the payoff to player II. Setting $V = 6$ and $C = 2$ produces a popular form of Prisoners' Dilemma.

Note that the values of V and C influence the behavior of Nash equilibrium in Hawk-Dove games. For Prisoners' Dilemma, there exists only one Nash equilibrium in terms of pure strategies. Indeed, the pure strategy profile (Hawk, Hawk) so strongly dominates other pure strategy profiles that extending the game to the mixed game fails to produce any new Nash equilibria [1]. As stated above, quantizing Prisoners' Dilemma a la Eisert at al. and using mixed quantum strategies gives rise to Nash equilibria that are superior to (Hawk, Hawk).

3. QUANTUM HAWK-DOVE GAMES

A Hawk-Dove game is typically quantized (properly) by identifying the set of outcomes

{(Dove, Dove), (Dove, Hawk), (Hawk, Dove), (Hawk, Hawk)}
$$=\{(D,D),(D,H),(H,D),(H,H)\}$$

of the game with an orthogonal basis $B = \{b_1, b_2, b_3, b_4\}$ of the state space of some appropriate quantum system, where the state space is a projective Hilbert space. This allows for the forming of quantum superpositions of the outcomes of the game. For the sake of notational simplicity, suppose the identification preserves order. It is important to note here that the pair of outcomes (H, D) and (D, H), the *best possible outcomes* in the game for player I and player II respectively, are identified with the basis elements b_3 and b_2, respectively.

A function F, referred to in the literature as a *quantization protocol* [2], now maps the pure quantum strategies of the players to a quantum superposition of the game's outcomes. Measurement produces a probability distribution over the outcomes from which the expected payoffs to the players as well as Nash equilibria are computed.

3.1 Nash Equilibrium

The state space of quantum superpositions is, among other things, an inner product space entertaining notions of angle and norm. As per the axioms of quantum mechanics, measurement is the orthogonal projection of a quantum superposition onto elements of an orthogonal basis of the state space. It now follows from elementary linear algebra that the smaller the angle between a quantum superposition and a basis element, the larger the norm of the measurement along that basis element. Moreover, quantum mechanics views the norm of the measurement of a quantum superposition along a basis element as the probability with which the quantum superposition projects onto that basis element. Therefore, a quantum superposition, upon measurement, will project with highest probability onto an element of an orthogonal basis with which it forms the smallest angle. Such a basis element is "closest" to a given quantum superposition.

More precise, let θ be the angle between a quantum superposition P and an element Q of an orthogonal basis of the state space. Then

$$\cos\theta = \left|P \cdot Q\right|^2 \qquad (1)$$

relates the angle between *P* and *Q* to the norm of the measurement of the quantum superposition along *Q*. It follows from Eq. (1) that the smaller the value of θ, the larger the norm of the measurement along *Q*. In the following discussion, this inverse relationship between the angel and the norm will be used interchangeably.

For quantum Hawk-Dove games, this mathematical structure translates into each player seeking a pure quantum strategy that will produce, via the quantization protocol, a quantum superposition that is closest to his best possible outcome, given the pure quantum strategy of the other player. In other words, each player will choose a pure quantum strategy that is the best reply to his opponent's choice of pure quantum strategy. If every player succeeds in finding such a pure quantum strategy, then the resulting pure quantum strategy profile is Nash equilibrium.

Again, more precisely, let player I use the pure quantum strategy *p* and let player II use the pure quantum strategy *q*. Suppose that the pair (*p*, *q*) is a pure quantum strategy Nash equilibrium in a quantum Hawk-Dove game. The quantization protocol *F* takes (*p*, *q*) to a quantum superposition

$$S_{(p,q)} = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \alpha_4 b_4 \qquad (2)$$

where $\alpha_i = \left|S_{(p,q)} \cdot b_1\right|^2$ are complex number satisfying

$$\sum_{i=1}^{4} \left|\alpha_i\right|^2 = 1 \qquad (3)$$

In Eq. (3), $\left|\alpha_i\right|^2$ is the norm of the measurement of $S_{(p,q)}$ along the *i*-th basis element, for $1 \leq i \leq 4$.

Now suppose player II switches his pure quantum strategy to *q**. The pure quantum strategy pair (*p*, *q**) is mapped by *F* to the quantum superposition $S_{\left(p,q^*\right)}$. Since *q* is the best reply to *p*, the Nash equilibrium pair (*p*, *q*) satisfies

$$\left|S_{(p,q)} \cdot b_2\right|^2 \geq \left|S_{\left(p,q^*\right)} \cdot b_2\right|^2 \qquad (4)$$

Similarly, if player I switches pure quantum strategy to *p**. The pure quantum strategy pair (*p**, *q*) is mapped by *F* to the quantum superposition $S_{\left(p^*,q\right)}$ and as Nash equilibrium (*p*, *q*) also satisfies

$$\left|S_{(p,q)} \cdot b_3\right|^2 \geq \left|S_{\left(p^*,q\right)} \cdot b_3\right|^2 \qquad (5)$$

4. CONCLUSIONS

The game theoretic notion of Nash equilibrium is characterized in terms of the geometry of the state space of quantum superpositions. In future, a more general set up will be developed for arbitrary two player, two strategy games and indeed possibly for *m* player, *n* strategy games. Potential applications of the geometric of Nash equilibrium of quantum games will be explored. For example, when the quantization protocol is a unitary operator, its image is a subspace of the state space of quantum superpositions. This guarantees the existence of an element in the image of the quantization protocol that is the best approximation, via the inner product, of the basis elements of the state space. A Fourier expansion of the basis elements can now be set up in terms of an orthogonal basis of the image of the quantization protocol. What this basis might be and what it might imply about the quantization protocol and how it might relate to the geometry of Nash equilibrium are open questions to be studied in future work.

5. REFERENCES

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