Animation Visualization for Vertex Coloring of Polyhedral Graphs

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ABSTRACT

Vertex coloring of a graph is the assignment of labels to the vertices of the graph so that adjacent vertices have different labels. In the case of polyhedral graphs, the chromatic number is 2, 3, or 4. Edge coloring problem and face coloring problem can be converted to vertex coloring problem for appropriate polyhedral graphs.

We have been developed an interactive learning system of polyhedra, based on graph operations and simulated elasticity potential method, mainly for educational purpose.

In this paper, we introduce a learning subsystem of vertex coloring, edge coloring and face coloring, based on minimum spanning tree and degenerated polyhedron, which is introduced in this paper.

Keywords: Vertex Coloring, Polyhedral Graph, Animation, Visualization, Interactivity

1. INTRODUCTION

Vertex coloring of a graph is the assignment of labels to the vertices of the graph so that adjacent vertices have different labels [1-3]. The 4-colour theorem proved by Appel and Haken in 1977, indicates that every planar graph is 4-colourable. Every polyhedral graph is 3-connected planar graph, according to the theorem by Steinitz. Therefore, it is also 4-colourable. Consequently, the chromatic number of a polyhedral graph is 2, 3, or 4. There are various coloring methods, for example, greedy coloring algorithm, sequential coloring algorithm, distributed algorithm, decentralized algorithm, and so on. Determination of 2-colourability is equivalent to testing bipartiteness, therefore, it is computable in linear time. However, in the case of more than 2 coloring, the computational complexity is known to be NP-complete, even for 3-colourability [4], and 4-colourability [5].

The author has been developed an interactive learning system of polyhedra, based on graph operations and simulated elasticity potential method, mainly for educational purpose [6-10]. By using this system, the user or the learner can make and handle various polyhedra, including Platonic solids, Archimedean solids [9], Kepler-Poinsot solids [7], fullerenes molecular structures, and geodesic dome constructions.

In this paper, we introduce a learning subsystem of interactive vertex coloring, edge coloring, and face coloring, based on minimum spanning tree and degenerated polyhedron. Vertex coloring of polyhedral graph itself is trivial in a mathematical sense, and it is not novel also in a practical sense. However, visibility and interactivity can be helpful for the user to understand intuitively the mathematical structure and the computational scheme, by visualizing the process of the calculation, and by allowing the user to contribute the computation.

2. POLYHEDRON MODELING SYTEM

In this section, we summarize the system of interactive modeling of polyhedra described in [6-10]. It consists of three subsystems: graph input subsystem, wire-frame subsystem, and polygon subsystem.

Graph Input Subsystem

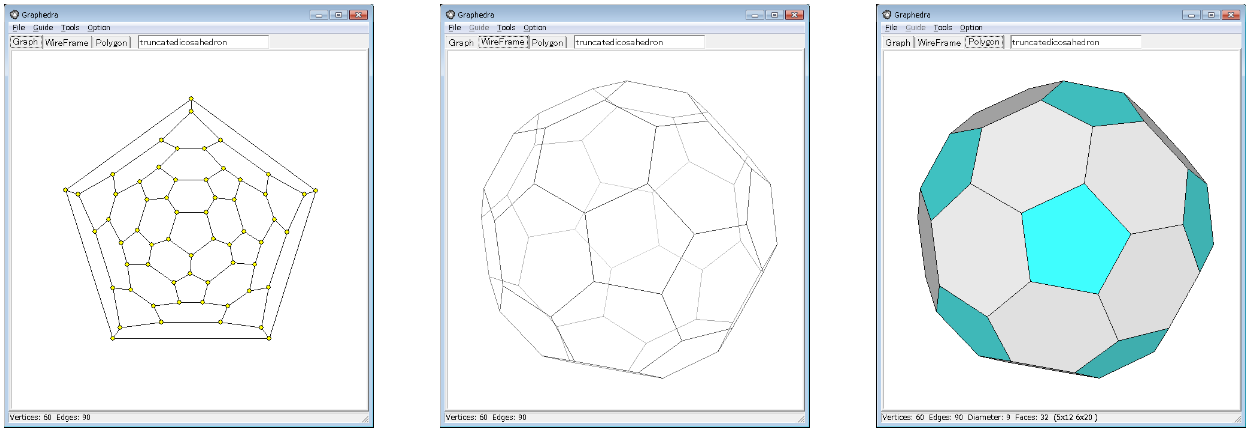
Figure 1(a) shows a screen shot of graph input subsystem, where a graph isomorphic to truncated icosahedron is drawn. The first step of the modeling of polyhedron is drawing a polyhedral graph isomorphic to the intended polyhedron. In the subsystem, vertex addition, vertex deletion, edge addition, and edge deletion are implemented as fundamental operations. Some additional utilities are also implemented such as grid lines, grid snapping, vertex coloring according to degrees, and so on.

Wire-Frame Subsystem

Figure 1(b) shows a screen shot of wire-frame subsystem. After constructing a polyhedral graph the next step is arranging vertices in 3D space with virtual springs and Hooke's law. Wire-frame polyhedron can be formed by controlling the natural length of virtual spring corresponding to three types of binary relations between pairs of vertices.

Polygon Subsystem

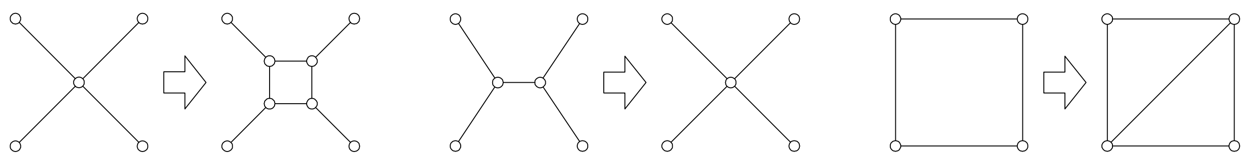
Figure 1(c) shows a screen shot of polygon subsystem. After arranging vertices in 3D space, the last step is detecting faces, selecting appropriate faces, and rendering the solid. Detecting n-polygon is equivalent to finding simple closed path with length n. Some additional utilities such as opening faces, meshed faces are implemented.



(a) (b) (c)

Figure 1. Screen shots of Interactive Polyhedron Modeling System.

(a) Graph input subsystem, (b) Wire-frame subsystem, and (c) Polygon subsystem.



(a) Vertex splitting (*vs*) (b) Edge contraction (*ec*) (c) Diagonal addition (*da*)

Figure 2. Three Graph Operations used in the wire-frame subsystem.

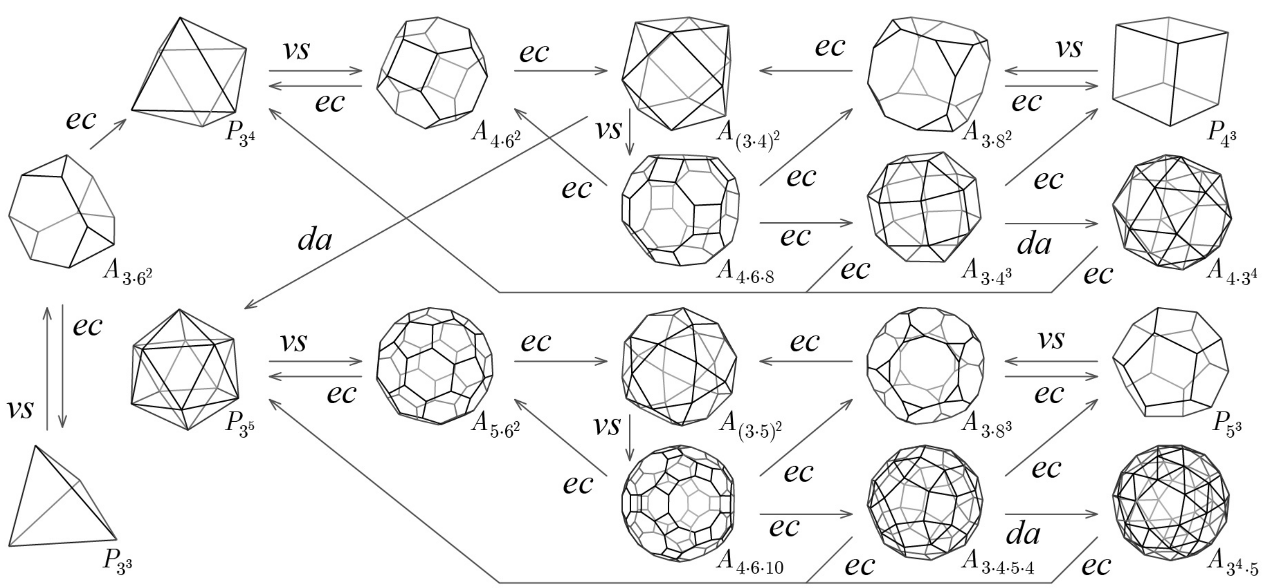


Figure 3. Relations of five Platonic graphs and thirteen Archimedean graphs using three graph operations.

Table 1. The list of regular polyhedra (Platonic solids) and semi-regular polyhedra (Archimedean solids).
v, *e*, and *f* stand for the numbers of vertices, edges and faces, respectively.
vc, *ec*, and *fc* are the chromatic numbers of vertex-coloring, edge-coloring and face-coloring.

Symbol	Name of polyhedron	*v*	*e*	*f*	*vc*	*ec*	*fc*
P_{3^3}	Tetrahedron	4	6	4	4	3	4
P_{4^3}	Cube	8	12	6	2	3	3
P_{3^4}	Octahedron	6	12	8	3	3	2
P_{5^3}	Dodecahedron	20	30	12	3	3	4
P_{3^5}	Icosahedron	12	30	20	4	3	3
$A_{(3\cdot4)^2}$	Cuboctahedron	12	24	14	3	3	2
$A_{4\cdot6\cdot10}$	Great Rhombicosidodecahedron	120	180	62	2	3	3
$A_{4\cdot6\cdot8}$	Great Rhombicuboctahedron	48	72	26	2	3	3
$A_{(3\cdot5)^2}$	Icosidodecahedron	30	60	32	3	3	2
$A_{3\cdot4\cdot5\cdot4}$	Small Rhombicosidodecahedron	60	120	62	3	3	2
$A_{3\cdot4^3}$	Small Rhombicuboctahedron	24	48	26	3	3	2
$A_{3^4\cdot4}$	Snub Cube	24	60	38	3	3	3
$A_{3^4\cdot5}$	Snub Dodecahedron	60	150	92	4	3	3
$A_{3\cdot8^2}$	Truncated Cube	24	36	14	3	3	4
$A_{3\cdot10^2}$	Truncated Dodecahedron	60	90	32	3	3	4
$A_{5\cdot6^2}$	Truncated Icosahedron	60	90	32	3	3	4
$A_{4\cdot6^2}$	Truncated Octahedron	24	36	14	2	3	3
$A_{3\cdot6^2}$	Truncated Tetrahedron	12	18	8	3	3	4

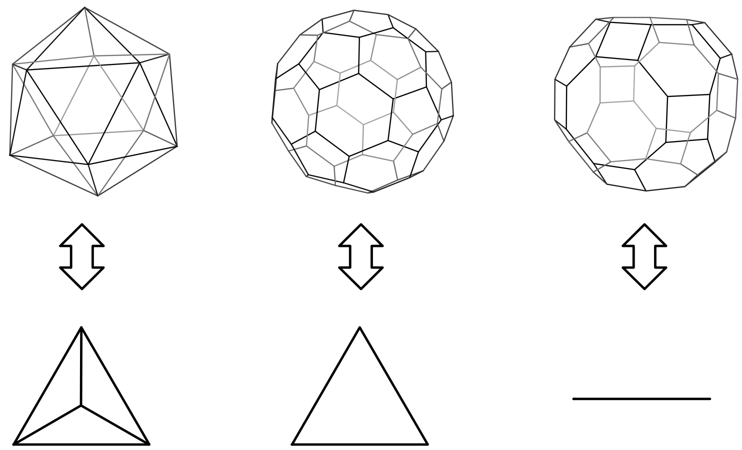
Graph Operation for Polyhedral Graph

Three graph operations are defined for polyhedral graphs: *vertex splitting* (*vs*), *edge contraction* (*ec*), and *diagonal addition* (*da*) (Figure 2) [9]. By these three operations, 5 regular polyhedra (Platonic solids) and 13 semi-regular polyhedra (Archimedean solids) are interconnected as shown in Figure 3. By using these operations, the user can model various polyhedra from one seed polyhedron.

3. ANIMATION VISUALIZATION OF VERTEX COLORING

Table 1 shows the complete list of regular polyhedra and semi-regular polyhedra. Symbols *v*, *e*, and *f* stand for the numbers of vertices, edges and faces. Symbols *vc*, *ec*, and *fc* are the chromatic numbers of vertex coloring, edge coloring and face coloring, respectively. Face coloring of a planar graph *G* is equivalent to vertex coloring of the dual of *G*. Edge coloring of a polyhedral graph *G* is equivalent to vertex coloring of the *ambo* of *G*. Ambo is one of Conway Polyhedron notations [10].

It is known that *k*-colorability and *k*-partiteness are equivalent for any graph. In the case of planar graph or polyhedral graph, the chromatic number, that is the maximum value of *k*, can be 2, 3 or 4. If we identify the vertices in each part of *k*-partite graph, one of polytopes is obtained among line segment (1-simplex), triangle (2-simplex), or tetrahedron (3-simplex). We call such polytopes obtained from polyhedra, *degenerated* polyhedra. Figure 4 shows three examples of degenerated polyhedra: icosahedron degenerated to tetrahedron, truncated icosahedron to triangle, and great rhombicuboctahedron to line segment.



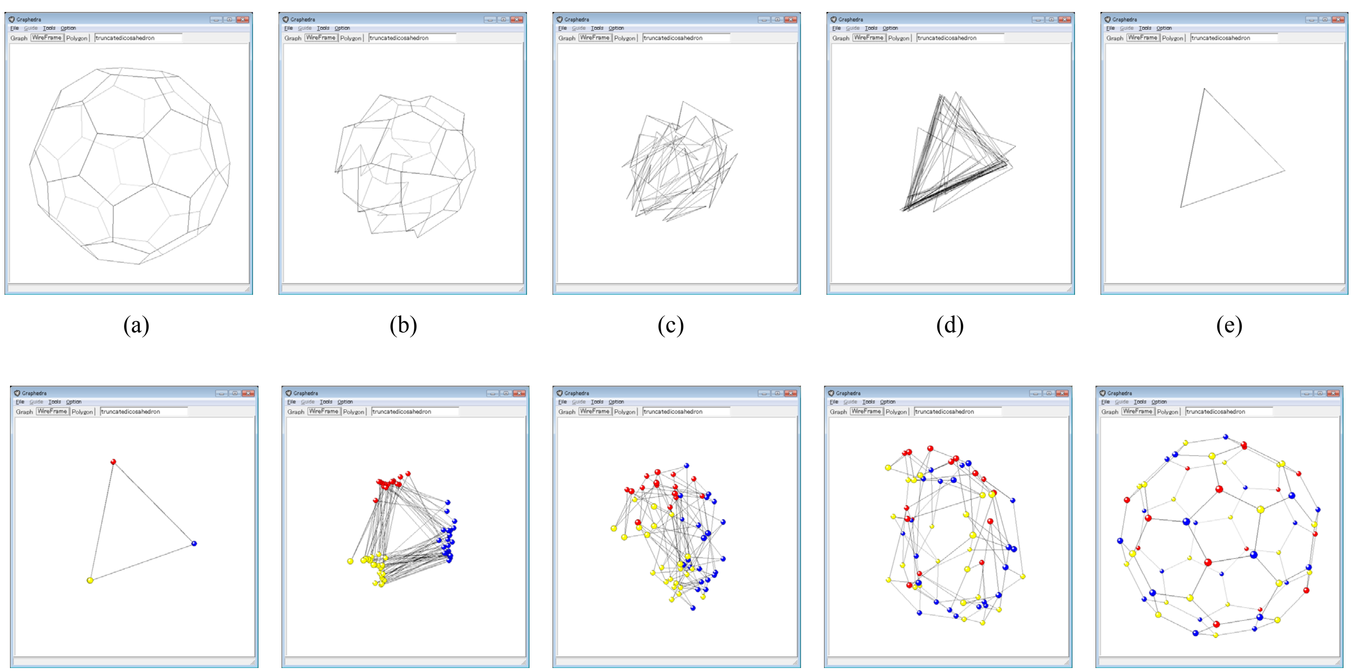
(a) Tetrahedron (b) Triangle (c) Line segment

Figure 4. Examples of degenerated polyhedra (polytopes).

When the user selects the menu “Degenerate” from a popup menu, the system tests the chromatic number of the graph G in the order of 2, 3, and 4 colorabilities. If the graph G is k-colorable, it is partitioned to k-partite graph. There is no edge within each part of the k-partite graph. For each part of G, a minimum spanning tree is generated, however the tree is not visible by the user. For each edge of the tree, virtual spring with zero-length is assigned. By applying the Hooke’s law, the graph G is degenerated to one of line segment, triangle, or tetrahedron, with animations (Figure 5 (a-e)).

After the user selects different color for each vertex of the degenerated polyhedron (Figure 5 (f)), when the user selects the menu “Release” from the popup menu, the polyhedron recovers the original shape with also animations (Figure 5 (g-j)).

Through the interactive operations, the user or learner can observe how the colors are assigned to the vertices so that adjacent vertices have different colors, and also understand unconsciously that k-colorable and k-partite are equivalent.



(a) (b) (c) (d) (e)

(f) (g) (h) (i) (j)

Figure 5. An example of animation-visualized vertex coloring (truncated icosahedron and triangle).

4. SUMMARY

In this paper, a vertex coloring system for polyhedral graph has been presented. It was developed as a subsystem of an interactive learning system of polyhedra, based on graph theory. A notion of degenerated polyhedron was introduced. It is known a polyhedral graph is 4-colorable, therefore, a polyhedral graph is degenerated to one of polytopes among a tetrahedron, a triangle, and a line segment. Through interactive operations, the learner can not only observe how the colors are assigned to the vertices, but also understand unconsciously that the notions of *k*-colorability and *k*-partiteness are equivalent.

We confirmed the effectiveness of our system through several actual lectures in under-graduate course. At present, a quantitative evaluation is progressing.

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