Inverse Problem In Optical Tomography Using Diffusion Approximation and Its Hopf-Cole Transformation

Taufiquar R. Khan
Department of Mathematical Sciences, Clemson University
Clemson, SC 29634-0975
khan@clemson.edu

ABSTRACT

In this paper, we derive the Hopf-Cole transformation to the diffusion approximation. We find the analytic solution to the one dimensional diffusion approximation and its Hopf-Cole transformation for a homogenous constant background medium. We demonstrate that for a homogenous constant background medium in one dimension, the Hopf-Cole transformation improves the stability of the inverse problem. We also derive a Green's function scaling of the higher dimensional diffusion approximation for an inhomogeneous background medium and discuss a two step reconstruction algorithm.

Keywords: Radiative transport, optical tomography, Hopf-Cole transformation, Green's function scaling, inverse problems.

1 INTRODUCTION

In the last decade research in the area of biomedical optics has flourished. In particular there has been considerable new development in bio-medical imaging using optical tomography [11, 3, 8, 18]. Optical tomography is a way to probe highly scattering media using low-energy visible or near infra-red light (NIR) and then to reconstruct images of these media. Light in the near-infrared range (wavelength from 700 to 1200 nm) penetrates tissue and interacts with it. The predominant effects are absorption and scattering [12, 14, 6]. We assume: given a set of measurements of transmitted light between pairs of points on the surface of an object, there exists a unique distribution of internal scatters and absorbers which would yield that set. The formation of an image for the optical properties of the tissue from a series of boundary measurements is the inverse problem of Optical Absorption and Scattering Tomography (OAST). The widely accepted photon transport model is the radiative transfer equation (RTE). The transport equation is an integro-differential equation for the radiance and has spatially dependent diffusion and absorption parameters as coefficients which are a priori unknown. Hence the problem is to infer from the measurements of the photon density on the boundary, the coefficients of absorption and diffusion in the tissue.

A low order diffusion approximation to the transport equation has been derived and studied in the last several years. This is an approximation to the transport equation by a parabolic differential equation in the time domain and by an elliptic differential equation in the frequency domain [3]. The diffusion approximation to the transport equation has been widely used to calculate photon migration in biological tissues [15]. The existing computational methods for the inverse problem for photon migration in biological tissues are almost exclusively based on the diffusion approximation [7].

It is well known that the diffusion based inverse problem in optical tomography is exponentially unstable [20, 3]. In order to understand the effect of Hopf-Cole transformation on the stability of the inverse problem, we derive the Hopf-Cole transformation to the diffusion approximation. We find the analytic solutions of the one dimension diffusion approximation and its Hopf-Cole transformation for a homogenous constant background medium. The outline of the paper is as follows. In section 2, we discuss the transport model and its diffusion approximation. In section 3, we derive the Hopf-Cole transformation to the diffusion approximation and find the analytic solution for a homogenous constant background medium in one dimension. In section 4, we derive the Green's function scaling of the diffusion approximation and discuss a two-step reconstruction algorithm. In section 5, we discuss our results and work in progress.

2 PHOTON TRANSPORT MODEL

In optical imaging, low-energy visible light is used to illuminate the biological tissue. The illumination of the tissue can be modelled as a photon transport phenomenon. The process is described by the most widely applied equation in optical imaging, the radiative transfer or transport equation (RTE) [9, 16]:

$$\frac{1}{c}\frac{\partial \phi}{\partial t}(\vec{r}, \hat{s}, t) + \hat{s}\cdot\nabla\phi(\vec{r}, \hat{s}, t) + \mu(\vec{r})\phi(\vec{r}, \hat{s}, t)$$
$$= D(\vec{r})\int_{S^{n-1}} \Theta(\hat{s}\cdot\hat{s}')\phi(\vec{r}, \hat{s}', t)d\hat{s}' + S_0(\vec{r}, \hat{s}, t) \quad (1)$$

together with the initial and boundary conditions,

$$\begin{aligned} \phi(\vec{r}, \hat{s}, 0) &= 0 \text{ in } \Omega\times S^{n-1} && (2)\\ \phi(\vec{r}, \hat{s}, t) &= 0 \text{ in } \partial\Omega\times S^{n-1}\times R^1 \ , && (3)\\ & \quad \hat{n}\cdot\hat{s} \le 0 \ , \ t \ge 0 \end{aligned}$$

which describes the change of radiance $\phi(\vec{r}, \hat{s}, t)$ of the photons at $\vec{r} \in \Omega \subset R^n$ travelling in the direction $\hat{s} \in S^{n-1}$, unit sphere in R^n, at time t. The parameters μ and D are the sought-for absorption and scattering tissue parameters, and c is the velocity of light. The function Θ is the scattering phase function characterizing the intensity of a wave incident in direction $\hat{s}'$ scattered in the direction $\hat{s}$. Simpler deterministic models can be derived from RTE by expanding the density ϕ, source S_0, and phase function Θ in spherical harmonics and retaining a limited number of terms [19, 5, 4, 3]. The simplest is the time dependent diffusion approximation P_0 (DA):

$$\frac{1}{c}\frac{\partial \phi_0}{\partial t}(\vec{r}, t) - \vec{\nabla}\cdot D(\vec{r})\vec{\nabla}\phi_0(\vec{r}) + \mu_a(\vec{r})\phi_0(\vec{r}) = S_0(\vec{r}) \quad (4)$$

together with initial and boundary conditions,

$$\begin{aligned} \phi_0(\vec{r}, 0) &= 0 \text{ in } \Omega\times S^{n-1} && (5)\\ \phi_0(\vec{r}, t) + 2D(\vec{r})\frac{\partial\phi_0}{\partial n}(\vec{r}, t) &= 0, \ \vec{r}\in\partial\Omega, && (6) \end{aligned}$$

where spherical harmonics to first order for the expansion of ϕ and zeroth order for the expansion of S_0 are used. The measurable quantity for the diffusion approximation is

$$g(\vec{r}, t) \quad = \quad -D(\vec{r})\frac{\partial\phi_0}{\partial n}(\vec{r}, t), \ \vec{r}\in\partial\Omega, \ t\ge 0. \quad (7)$$

Frequency-domain diffusion approximation can easily be obtained by Fourier transforming the time-domain equation. The frequency domain analog of equation (4) is given by

$$-\vec{\nabla}\cdot D(\vec{r})\vec{\nabla}\phi(\vec{r}, \omega) + \left(\mu_a(\vec{r}) + \frac{i\omega}{c}\right)\phi(\vec{r}, \omega) = S_0, \quad (8)$$

where we made use of the relation

$$\frac{\partial}{\partial t} \equiv i\omega.$$

The frequency domain DE is elliptic where the time-domain DE is parabolic, an important distinction for numerical solutions. Furthermore, the diffusion approximation to the radiative transfer model can be written in the time independent (dc) case as [3],

$$-\vec{\nabla}\cdot D(\vec{r})\vec{\nabla}\phi(\vec{r}) + \mu_a(\vec{r})\phi(\vec{r}) = S_0(\vec{r}). \quad (9)$$

The associated boundary condition is

$$\hat{n}\cdot D(\vec{\zeta})\vec{\nabla}\phi(\vec{\zeta}) + \alpha\phi(\vec{\zeta}) = 0, \quad (10)$$

where $\vec{\zeta}$ is the position on the boundary, and $\hat{n}$ is the unit vector normal to the boundary and α is a calibration constant (which can be determined empirically by matching the forward calculations to well controlled experiments).

If we let Ω be the domain under consideration with surface $\partial\Omega$, we can define the forward problem as: given sources S_0 in Ω and q in Q, a vector of model parameters, for example the coefficient of diffusion D and the coefficient of absorption μ_a (i.e. $q = (D, \mu_a)$) that belongs to a parameter set Q, find the data ϕ_m on $\partial\Omega$ and the inverse problem as: given data ϕ_m on $\partial\Omega$ find q. We can recast the forward problem in an abstract setting as the following parameter dependent equation:

$$-\vec{\nabla}\cdot D(\vec{r})\vec{\nabla}\phi(\vec{r}; q) + \mu_a(\vec{r})\phi(\vec{r}; q) \quad = \quad S_0(\vec{r}),$$

where $\vec{r}$ is in Ω, $\phi(q)$ is in an appropriate abstract space H, and S_0 represents a source or a forcing distribution. In general, measurement of $\phi(q)$ may not be possible, only some observable part $\mathcal{C}\phi(q)$ of the actual state $\phi(q)$ may be measured. In this abstract setting, the objective of the inverse or parameter estimation problem is to choose a parameter q^* in Q, that minimizes an error criterion or cost functional $J(\phi(q), \mathcal{C}\phi(q), q)$ over all possible q in Q subject to $J(q)$ satisfying the diffusion approximation. A typical observation operator is,

$$\mathcal{C}\phi(q) \quad = \quad \{\phi(\zeta_i, q)\}_{i=1}^N$$

where ζ_i is in $\partial\Omega$ and N is the number of measurements. A typical cost functional J_λ is given as,

$$J_\lambda(q) \quad = \quad \sum_{i=1}^N |\phi(\zeta_i, q) - z_i|^2 + \lambda\|q - q_0\|^2$$

where z_i is the measured photon density at the boundary and λ is the regularization parameter. Now composing $\phi(q)$ and $\mathcal{C}\phi(q)$ we obtain the parameter-to-output mapping: $T[\phi] = \mathcal{C}\phi$. This is the nonlinear mapping of diffusion based optical tomography in abstract setting.

For example, in one dimension with $\Omega = [0, l]$, the diffusion approximation with constant background is the Strum-Louiville equation:

$$-\nabla^2\phi + q^2\phi = 0 \quad (11)$$

where $q^2 = \mu_a/D$ is constant, with the Rubin boundary condition:

$$\begin{aligned} \phi(0) - \alpha\nabla\phi(0) &= 0 \\ \phi(l) + \alpha\nabla\phi(l) &= 0. \end{aligned} \tag{12}$$

The inverse problem is to estimate the scalar q from the photon density z measured at $x = 0$ or $x = l$.

3 HOPF-COLE TRANSFORMATION IN 1-D

In this section, we derive the Hopf-Cole [13, 17] transformation to the diffusion approximation. We begin by transforming ϕ:

$$\nabla\psi = D\nabla(ln(\phi)) \tag{13}$$

which is the Hopf-Cole transformation and converts equation (11) and (12) to:

$$-\nabla^2\psi - \frac{|\nabla\psi|^2}{D} + q^2 D = 0 \tag{14}$$

where $q^2 = \mu_a/D$ is constant, with the Rubin boundary condition:

$$\begin{aligned} \alpha\nabla\psi(0) &= D \\ \alpha\nabla\psi(l) &= -D. \end{aligned} \tag{15}$$

The problem (11) and (12) can be solved analytically and the solution for $x < \eta$ is:

$$\phi(x, \eta; q) = \frac{(e^{q\eta} - \gamma e^{-q\eta})(e^{qx} - \beta e^{-qx})}{2qD(\beta - \gamma)} \tag{16}$$

where

$$\begin{aligned} \gamma &= \frac{e^{2lq}}{\beta} \\ \beta &= \frac{1 - \alpha q}{1 + \alpha q}. \end{aligned}$$

If we measure the solution at $x = 0$, then the inverse problem is to estimate q from the data z measured at $x = 0$. Therefore the parameter to output map is given by,

$$\begin{aligned} Tq &= C\phi(x, \eta; q) \\ &= \phi(0, \eta; q) \\ &= \frac{(e^{q\eta} - \gamma e^{-q\eta})(1 + \beta)}{2(\beta - \gamma)} \end{aligned} \tag{17}$$

which is a nonlinear function of the parameter q as expected. Similarly the solution of (14) and (15)is:

$$\psi(x; q) = Dln(\beta - e^{2qx}) - qDx \tag{18}$$

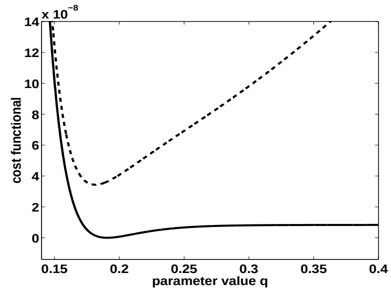

Figure 1: Parameter Estimation and Regularization

and for a measurement at $x = 0$ the parameter to output map is given by,

$$\begin{aligned} \tilde{T}q &= C\psi(x; q) \\ &= \psi(0; q) \\ &= Dln(\beta - 1). \end{aligned} \tag{19}$$

In Figure 1, we plot $J_\lambda(q)$ for the 1D diffusion approximation with Rubin boundary conditions for a homogenous background medium with $\mu_a = 0.012\text{mm}^{-1}$ and $D = 0.33$mm. This is the simulation of the 1D diffusion approximation on the interval $(0, 43.0)$ with $q = \sqrt{\mu_a/D}$. We computed cost functional $J_\lambda(q) = |\phi(0; q_0) - \phi(0; q)| + \lambda\|q\|^2$ for $q_0 = 0.1907mm^{-1}$ (corresponding to $\mu_a = 0.012\text{mm}^{-1}$ and $D = 0.33$mm) over a range of q starting from 0.14 to 0.4. The solid curve represents J without regularization ($\lambda = 0$) and the broken curve represents J_λ with regularization parameter $\lambda = 10^{-6}$. From Figure 1, it is clear that without regularization ($\lambda = 0$) the function J is rather insensitive to parameter $q = \sqrt{\mu_a/D}$ (i.e. the numerical method starting with an overestimate of the true parameter is bound to fail). But with regularization ($\lambda = 10^{-6}$), J_λ is more convex. We note here that the regularization has changed the problem so that we are solving for a minimum q_λ that is no longer the same as the problem without regularization, mainly q_0. In Figure 2, we plot $J(q)$ for the Hopf-Cole transformation of the diffusion approximation. Similar to Figure 1, the solid curve represents J_λ without regularization ($\lambda = 0$) and the broken curve represents J_λ with regularization ($\lambda = 1.0$). From comparison of Figure 1 and 2, it is clear that for a homogenous constant background medium in one dimension, Hopf-Cole transformation makes the cost functional J_λ more convex with respect to the parameter q. The transformation also changes the scale of the solution as expected from equation (13). From Figure 2, it is also clear that regularization (broken line) does not improve the convexity of the cost functional (solid line) and that the function J is more sensitive to parameter $q = \sqrt{\mu_a/D}$ (i.e. the numerical method starting anywhere should converge to the true parameter and hence improves the stability of the inverse problem).

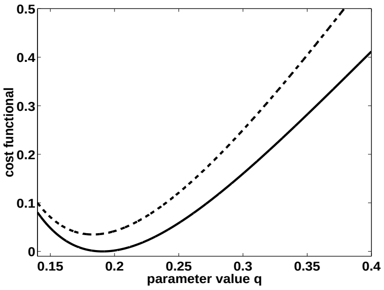

Figure 2: Parameter Estimation Using Hopf-Cole Transformation

4 GREEN'S FUNCTION SCALING IN 2-D

In this section, we derive a Green's function scaling of the diffusion approximation. We begin by transforming ϕ as,

$$\phi = \Pi_\kappa(\Psi), \tag{20}$$

where Π_κ is an invertible, in general nonlinear, transformation that is twice continuously differentiable and κ is a scaling constant. The goal is to find a suitable Π_κ to enhance the resolution so that Ψ is more uniform than ϕ in Ω. This also can be thought of as preconditioning similar to the idea of boosting in distorted born approximation [21, 10]. For example, if $\Pi_\kappa(\Psi) = G\Psi$ where $G(\vec{r}, \vec{r}_0)$ represents the Green's function for the diffusion approximation (9), then the diffusion approximation (9) transforms into:

$$-\nabla \cdot (a\nabla\Psi) - b \cdot \nabla\Psi = S_0' \tag{21}$$

where $S_0' = S - \phi(\vec{r}_0)$, $a = DG(\vec{r}, \vec{r}_0)$, and $b = D\nabla G(\vec{r}, \vec{r}_0)$. The associated boundary condition transforms into: $\hat{n} \cdot \left(a(\vec{\zeta})\nabla\Psi(\vec{\zeta})\right) = 0$ where $\vec{\zeta} \in \partial\Omega$. In what follows, we will refer to equation (21) as the scaled diffusion approximation. There are several remarks in order. In the unscaled version, both μ_a and D appear explicitly while in the scaled version, only D appears explicitly. Of course, in the scaled version, the dependence of μ_a and D is hidden in the Green's function G. One can take advantage of this implicit-explicit dependence and consider a two-step reconstruction algorithm analogous to distorted born approximation [21, 10] discussed below. In Figure 3, unscaled and scaled solution for D and μ_a distribution are shown in Figure 3cd respectively, using the finite element method. The scaled solution is obtained using the simplified free space Green's function as an approximation for G, where constant background values of the diffusion and absorption are used [1, 2]. The change in uniformity due to scaling is evident from looking at Figure 3.

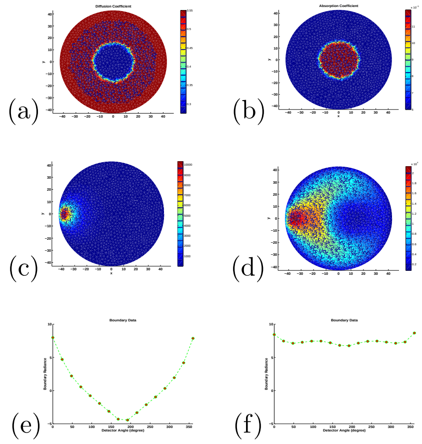

Figure 3: Diffusion coefficient (a), absorption coefficient (b), unscaled solution (c), scaled solution (d), unscaled boundary data in log scale (e), and scaled boundary data in log scale (f).

Both Hopf-Cole and Green's function scaling can be best studied in higher dimension using a two-step reconstruction algorithm analogous to the distorted born approximation. Given an initial guess $(D^{(0)}, \mu_a^{(0)})$, the inverse problem will be solved by optimizing J over μ_a, only using a forward solver for Equation (9) and getting an update $\mu_a^{(1)}$. Using this new set of parameters $(D^{(0)}, \mu_a^{(1)})$, the approximation to the Green's function $G(D^{(0)}, \mu^{(1)})$ will be computed. This $G(D^{(0)}, \mu^{(1)})$ will be used to solve the optimization problem J over D using the forward solver for Equation (21) and obtaining an update for $D^{(1)}$. This process will then continue until a tolerance is achieved.

5 CONCLUSION

We have shown that for a homogenous constant background medium in one dimension, the Hopf-Cole transformation convexifies the cost functional J. However the Hopf-Cole transformation involves solving a more complicated forward equation (it transforms the linear problem into a nonlinear one, see equation 14). Nevertheless the transformation improves the stability of the inverse problem which is important because the diffusion based inverse problem is severely ill-posed. A typical reconstruction algorithm requires regularization using Newton's method, Kaczmarz method, conjugate gradient method, and so forth. Moreover, regularization essentially changes the problem and may or may not be the one that is desired physically and also the problem of finding the right regularization may be very difficult. Furthermore, the use of appropriate regularization technique is critical, which should warrant that q_λ approaches the true solution q as λ approaches

0. The Hopf-Cole transformation leads to a complicated nonlinear equation but it improves the stability of the inverse problem without any regularization. We also investigated a Green's function scaling of the diffusion approximation in higher dimension. We demonstrated that the Green's function scaling can be used to enhance resolution by preconditioning the solution. We are currently investigating the effect of the Hopf-Cole/Green's function transformations on the stability of the inverse problem for an inhomogenous background medium in higher dimension using the two step reconstruction algorithm discussed.

References

[1] S.R. Arridge. Photon-measurment density functions. part 1: Analytic forms. *Applied Optics*, 34(33):7395–7401, 1995.

[2] S.R. Arridge. Photon-measurment density functions. part 2: Finite-element method calculations. *Applied Optics*, 34(34):8026–8307, 1995.

[3] S.R. Arridge. Optical tomography in medical imaging: Topical review. *Inverse Problems*, 15:R41–R93, 1999.

[4] S.R. Arridge and J.C. Hebden. Optical imaging in medicine: 2. modelling and reconstruction. *Phys. Med. Biol.*, 42:841–853, 1997.

[5] H. Bremmer. Random volume scattering. *Radiat. Sci. J. Res.*, 680:967–981, 1964.

[6] B. Chance and R.R. Alfano, editors. *Photon migration and imaging in random media and tissues*, volume 1888. SPIE, 1993.

[7] B. Chance and R.R. Alfano, editors. *Optical tomography, photon migration, and spectroscopy of tissue and model media: theory, human studies, and instrumentations, part 1 and 2*, volume 2389. SPIE, 1995.

[8] B. Chance, C.E. Cooper, D.T. Delpy, and E.O.R. Reynolds. Near-infrared spectroscopy and imaging of living systems. *Phil. Trans. R. Soc. Lond. B*, 352:643–648, 1997.

[9] R. Chandrasekhar. *Radiation Transfer.* Oxford, Clarendon, 1950.

[10] W.C. Chew and Y.M. Wang. Reconstruction of two-dimensional permittivity distribution using the distorted born iterative method. *IEEE Transactions on Medical Imaging*, 9(2):218–225, 1990.

[11] National Research Council. *Mathematics and Physics of Emerging Biomedical Imaging.* National Academy Press, Washington DC, 1996.

[12] D.T. Delphy and M. Cope. Quantification in tissue near-infrared spectroscopy. *Phil. Trans. R. Soc. Lond. B*, 352:649–659, 1997.

[13] Lawrence C. Evans. *Partial Differential Equations.* American Mathematical Society, Providence, 1998.

[14] J. Hebden, S.R. Arridge, and D.T. Delphy. Optical imaging in medicine: 1. experimental techniques. *Phys. Med. Biol.*, 42:825–840, 1997.

[15] A. Hielscher, R. Alcouffe, and R. Barbour. Comparison of finite-difference transport and diffusion calculations for photon migration in homogenous and heterogenous tissues. *Phys. Med. Biol.*, 42:1285–1302, 1998.

[16] A. Ishimaru. *Single Scattering and Transport Theory (Wave Propogation and Scattering in Random Media I.* Academic, New York, 1978.

[17] K. T. Joseph and A.S. Vasudeva Murthy. Hopf-cole transformation to some systems of partial differential equations. *Nonlinear Differ. Equ. Appl.*, 8:173–193, 2001.

[18] M. Klibanov and T.R. Lucas. Numerical solution of a parabolic inverse problem in optical tomography using experimental data. *SIAM J. Appl. Math.*, 59(5):1763–1789, 1999.

[19] H.W. Lewis. Multiple scattering in an infinite medium. *Phys. Rev.*, 78:526–529, 1950.

[20] F. Natterer. *Mathematical Methods in Image Reconstruction.* SIAM Monographs on Mathematical Modeling and Computation, 2001.

[21] Y.M. Wang and W.C. Chew. Reconstruction of permittivity distribution using the born iteration and distorted born iterative methods for geophyscial applications. *Int. Geoscience and Remote Sensing Symposium, College Park*, pages 379–381, 1990.