Monte Carlo Simulation of an American Option

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ABSTRACT

We implement gradient estimation techniques for sensitivity analysis of option pricing which can be efficiently employed in Monte Carlo simulation. Using these techniques we can simultaneously obtain an estimate of the option value together with the estimates of sensitivities of the option value to various parameters of the model. After deriving the gradient estimates we incorporate them in an iterative stochastic approximation algorithm for pricing an option with early exercise features. We illustrate the procedure using an example of an American call option with a single dividend that is analytically tractable. In particular we incorporate estimates for the gradient with respect to the early exercise threshold level.

Keywords: Black-Scholes model, Simulation, Perturbation Analysis, Gradient, Option Pricing.

1. INTRODUCTION

The increasing complexity of the underlying dynamics in option pricing which violates the assumptions of simpler analytical models has necessitated the use of other models. One popular method that practitioners often result to is the Monte Carlo simulation. Boyle [2] was among the first to propose using Monte Carlo simulation to study option pricing. Other researchers who have employed Monte Carlo simulation for analyzing option market include Johnson and Shanno [11] and Hull and White [10]. Boyle et al. [3] give an overview of pricing using Monte Carlo simulation. Our goal here will be to employ efficient gradient estimation techniques known as perturbation analysis (PA) in Monte Carlo simulation. PA techniques are described in more details in Cao [4], Ho and Cao [8] and Glasserman [7]. Numerical results have shown that these estimates are usually superior to the "brute force" method of finite difference. After obtaining the PA estimates we incorporate them in a stochastic approximation algorithm resulting in what is known as a Robbins-Monro-like algorithm (see Pflug [13] for a general discussion).

We now briefly provide some basic background information on option pricing. Basic references for derivative pricing include Hull [9], Gibson [6], and Cox and Rubinstein [5]. There are two basic types of options: a $call$ and a put. A call option is the contract right to buy a specified amount of an asset at a fixed price on or before the given date. A put option, on the other hand, is identical except that it is the right to sell the asset at the given price on or before the given date. If the option purchaser acts upon this right to buy, he or she is $exercising$ the right, and the fixed price of the transaction is called the $strike$ price. The last date on which the option may be exercised is called the $expiration$ date or the $maturity$ date. The difference in the legal exercising time of the options results in many different options, such as the $European$ option and the $American$ option. An American option allows the holder to exercise the right at any time before and including the expiration date, whereas a European option restricts the right only to expiration date and not before. The payoff of an option is the non-negative difference between the strike price and the asset price at exercise for a put option or in the case of a call option, the non-negative difference between asset price and strike price. A call option is said to be $in-the-money$ when its asset price is above the strike price; otherwise it is said to be $out-of-the-money$. A put option, on the other hand, is in-the-money when the asset price is below the strike price; otherwise it is out-of-the money.

The following notation will be used throughout the paper:

S_t= stock price at time t,
S_0= initial stock asset,
r= annualized risk-less interest rate (compounded continuously),
σ= volatility of the stock price,
μ= drift or other mean-related parameter of the underlying stock,
K= strike price of the option contract,
T= expiration date of the option contract,
J_T = the net present value return of the option on its expiration.

We will take the "present time" as time 0. Except for S_t and J_T which are random variables, we will assume that

the rest are constants.
In general to value an option using simulation one would:

1. Simulate the stock price dynamics, S_t, where $t \in [0, T]$;

2. Estimate $e^{-r\tau} E\left[f(S_t, 0 \le t \le \tau)\right]$,
where $\tau \in [0, T]$ is an optimal stopping time and f is a payoff function of the form:

European call : $(S_T - K)^+$,
American call : $(S_\tau - K)^+$,
Asian call : $(T^{-1} \int_0^T S_t dt - K)^+$,
Upper (knockout): $(S_T - K)^+ \mathbf{I}\{S_t < U, 0 \le t \le T\}$,
Double barrier (knockout) : $(S_T - K)^+ \mathbf{I}\{L < S_t < U, 0 \le t \le T\}$,
Lookback (strike) : $(S_T - \min\{S_t, 0 \le t \le T\})^+$.

where $\mathbf{I}\{\cdot\}$ is the indicator function.

2. GRADIENT ESTIMATION TECHNIQUES

The gradient estimation problem involves estimating the derivative of a deterministic function, in this case $J_T(\theta)$, with respect to the parameter θ. We will derive gradient estimates of the option value with respect to the early exercise threshold level. The commonly used gradient estimation techniques are:

1. Likelihood Ratio Method (LRM) (see Rubinstein and Shapiro [15]),

2. Perturbation Analysis (PA) which includes:

a) Infinitesimal Perturbation Analysis (IPA) (see Ho and Cao [8]),

b) Smoothed Perturbation Analysis (SPA).

In this paper we will only consider PA techniques.

Infinitesimal Perturbation Analysis
Infinitesimal perturbation analysis (IPA) is simply the sample path derivative of the performance measure of interest, $J_T(\theta)$ in this case. It is defined by

$$\frac{\partial J_T}{\partial \theta} = \lim_{\epsilon \to 0} \frac{J_T(\theta + \epsilon) - J_T(\theta)}{\epsilon} \quad \text{w.p. 1.} \tag{1}$$

For IPA estimator to be an unbiased gradient estimator, we need

$$\frac{\partial E\left[J_T\right]}{\partial \theta} = E\left[\frac{\partial J_T}{\partial \theta}\right]. \tag{2}$$

That is to say, unbiasedness reduces to the mathematical question of an interchange of expectation and limit and whose validity is checked by the dominated convergence theorem (see Royden [14]).

Dominated Convergence Theorem.
If $\lim_{h\to 0} g_h = g$ w.p.1 and there exist an $\epsilon > 0$ such that $|g_h| \le K$ w.p.1 for every $h \in [0, \epsilon]$ with $E[K] < \infty$ and K independent of θ, then $\lim_{h\to 0} E[g_h] = E[g]$.

If $J_T(\theta)$ can be shown to be continuous and piecewise differentiable on Θ w.p.1, which is usually accomplished by the help of the generalized mean value theorem, then the continuity of $J_T(\theta)$ automatically satisfies unbiasedness via the dominated convergence theorem.

Smoothed Perturbation Analysis
Suppose $J_T(\theta)$ is biased, that is to say the interchange given by Eq. (2) does not hold, generally because the sample performance function is not "smooth" enough. IPA technique then fails but an extension of it known as smoothed perturbation analysis (SPA), composed of an IPA component and a conditional component, helps remedy this problem. Using the conditional expectation operator, SPA provides a "smoothing" function. By conditioning on an appropriately chosen set of random variables, we form the conditional Monte Carlo estimator:

$$g_h = E\left[\left.\frac{J(\theta + h) - J(\theta)}{h}\right| Z\right], \tag{3}$$

and hope to get an interchange of expectation and limit.

3. AN AMERICAN CALL ON A SINGLE DIVIDENT-PAYING STOCK

Consider an American call option on a stock that distributes a dividend of amount D at time t_1, i.e., there is a single dividend payable during the lifetime of the contract $[0, T]$. We denote by τ_1 the time until the ex-dividend point and by τ_2 the time from the ex-dividend point to the expiration date. We assume that after the ex-dividend the stock price drops by the dividend amount, i.e., $S_{t_1^+} = S_{t_1^-} - D$, where D is assumed. We know that an American call option can be exercised at any time before the expiration date T. However, in this case where the stock pays a dividend, it is well known that the option should only be exercised $-$ if at all $-$ right before an ex-dividend date or at the expiration date. We assume that the following policy is adopted. There is an exercise threshold level $s(\ge K)$ such that if $S_{t_1^-} > s$, the option is exercised. We wish to obtain an optimal threshold s in order to maximize the expected option payoff. The sample performance is given by

$$J_T = e^{-rT} \hat{J}_T \tag{4}$$

where, $\hat{J}_T$, the undiscounted value of the option payoff is given by:

$$\begin{aligned}\hat{J}_T &= \mathbf{I}\{S_{t_1^-} > s\}(S_{t_1^-} - K)e^{r(T-t_1)} \\ &+ \mathbf{I}\{S_{t_1^-} \le s\}(S_T - K)^+ \end{aligned} \quad (5)$$

where $S_{t_1^-} = h(Z_1; \hat{S}_0, \tau_1, \mu, \sigma) + D$, $S_T = h(Z_2; S_{t_1^-} - D, \tau_2, \mu, \sigma)$, $\hat{S}_0 = S_0 - De^{-r\tau_1}$, Z_1 and Z_2 are two random variables with c.d.f. F_1 and F_2 and p.d.f. f_1 and f_2, respectively, and the stock price changes according to $h(Z; S, t, \mu, \sigma)$. For example, for the Black-Scholes log-normal distribution $h(Z; S, t, r, \sigma) = Se^{(r-\sigma^2/2)t+\sigma\sqrt{t}Z}$ where Z is a standard normal random variable. We are interested in estimating $\partial E[J_T]/\partial\theta$. S_t is not almost surely continuous with respect to its parameters, because there is a jump at the ex-dividend point. Hence the IPA estimator is biased, and so we result to SPA, with the estimator given by

$$\left(\frac{\partial J_T}{\partial\theta}\right)_{PA} = e^{-rT}\left[\left(\frac{\partial \hat{J}_T}{\partial\theta}\right)_{PA} - \hat{J}_T\frac{\partial}{\partial\theta}(rT)\right], \quad (6)$$

where

$$\begin{aligned}\left(\frac{\partial \hat{J}_T}{\partial\theta}\right)_{PA} &= \frac{\partial h^{-1}(y^*)}{\partial\theta} f_1(h^{-1}(y^*)) \\ &* \ (E[\hat{J}_T | S_{t_1^-} = s^-] - E[\hat{J}_T | S_{t_1^-} = s^+]) \\ &+ \ \ \mathbf{I}\{S_{t_1^-} > s\}\frac{\partial}{\partial\theta}\left[(S_{t_1^-} - K)e^{r(T-t_1)}\right] \\ &+ \ \mathbf{I}\{S_{t_1^-} \le s\}\frac{\partial}{\partial\theta}(S_T - K)^+ \end{aligned} \quad (7)$$

where, omitting the explicit display of μ and σ for simplification purposes, we define

$$\begin{aligned} E[\hat{J}_T | S_{t_1^-} = s^-] &= E[(S_T - K)^+ | S_{t_1^-} = s^-] \\ &= E[(h(Z_2; s - D, \tau_2) - K)^+], \quad (8) \\ E[\hat{J}_T | S_{t_1^-} = s^+] &= (s - K)e^{r(T-t_1)}, \quad (9) \\ y^* &= (s - D; \hat{S}_0, \tau_1). \quad (10) \end{aligned}$$

4. EXAMPLE

We illustrate the estimators with an example where we set θ to be s, the early exercise threshold level. We assume that the stock price follows the Black-Scholes log-normal distribution where $f_1(x) = f_2(x) = e^{-x^2/2}/\sqrt{2\pi}$. The inverse is given by $h^{-1}(y; S, t, r, \sigma) = (\ln(y/S) - (r - \sigma^2/2)t)/(\sigma\sqrt{t})$, so we have

$$h^{-1}(y^*) = \frac{1}{\sigma\sqrt{\tau_1}}\left(\ln\frac{s-D}{\hat{S}_0} - (r - \sigma^2/2)\tau_1\right), \quad (11)$$

and

$$\begin{aligned} \frac{\partial h^{-1}(y^*)}{\partial s} &= \frac{1}{(s-D)\sigma\sqrt{\tau_1}}; \quad (12) \\ \frac{\partial}{\partial s}[(S_{t_1^-} - K)e^{r\tau_2}] &= 0; \quad (13) \\ \frac{\partial}{\partial s}(S_T - K)^+ &= 0. \quad (14) \end{aligned}$$

5. STOCHASTIC OPTIMIZATION

We now formulate the problem as an optimization problem. The generic form of an optimization problem is defined as

$$\min_{\theta\in\Theta} g(\theta), \quad (15)$$

and the general form of a stochastic approximation algorithm is given by (see Kushner and Yin [12])

$$\theta_{n+1} = \Pi_\Theta(\theta_n - a_n\widehat{\nabla}g(\theta_n)), \quad (16)$$

where $\widehat{\nabla}g(\theta_n)$ is an estimate of the gradient from iteration n, Π_Θ is a projection onto the controllable set of parameters Θ, and a_n is a positive sequence of step sizes satisfying $\sum_1^\infty a_n = \infty$ and $\sum_1^\infty a_n^2 < \infty$.

Our option pricing problem can be viewed as an optimization problem where the option value is the point at which the expected return, $E(J_T(\theta))$, is maximized with respect to s, the early exercise threshold level. Since this is a maximization problem, the stochastic approximation algorithm given by Eq. (16) takes the positive version of the recursion, i.e.,

$$\theta_{n+1} = \Pi_\Theta(\theta_n + a_n\widehat{\nabla}g(\theta_n)), \quad (17)$$

and using Eqs. (6)-(10) we get:

$$\begin{aligned}\widehat{\nabla}g(\theta_n) &= e^{-rT}\frac{\partial h^{-1}(y^*)}{\partial s} f_1(h^{-1}(y^*)) \\ &* \ (E[(h(Z_2; s - D, \tau_2) - K)^+] \\ &- (s - K)e^{r(T-t_1)}), \end{aligned} \quad (18)$$

where the last two terms of Eq. (7) are zero as portrayed by Eqs. (13) and (14). For the example given in section 4, Eq. (18) becomes:

$$\begin{aligned}\widehat{\nabla} g(\theta_n) &= \frac{e^{-rT}}{(s-D)\sigma\sqrt{\tau_1}\sqrt{2\pi}} \\ &* \ exp-\frac{1}{2}\left[\frac{1}{\sigma\sqrt{\tau_1}}\left(\ln\frac{s-D}{\hat{S}_0}-(r-\frac{\sigma^2}{2})\tau_1\right)\right]^2 \\ &* \ \Big(E[(s-D)e^{(r-\sigma^2/2)\tau_2+\sigma\sqrt{\tau_2}Z_2}-K)^+] \\ &- \ (s-K)e^{r(T-t_1)}\Big). \qquad (19)\end{aligned}$$

6. NUMERICAL RESULTS

For the example given above we used the following setup. We took Θ to be the set of positive real numbers. In the stochastic approximation algorithm given by Eq. (17), we used the harmonic series, where $a_n = a/n$ and we took a to be 100 and n the number of simulation iterations to be 10,000. After the 10,000 iterations, an additional 10,000 independent replications were used to estimate the expected payoff. Initially the exercise threshold level was set to be the same as the strike price (50 in this case). We considered a fixed observation length of 100, but did not consider any stopping rule since we were more interested in tracking the improvement of the algorithm, which could be done easily for this analytically tractable example. We ran three sets of simulations corresponding to three dividend amounts of 0.5, 1.0 and 1.5. Each set consisted of 10 different initial threshold levels from 50 to 59 consecutively. The other parameters were kept constant at the following values:

$$K = S_0 = 50, r = 0.10,$$
$$\sigma = 0.3, (\tau_1, \tau_2) = (60, 30).$$

The three tables (one for each dividend amount) give the values of expected option payoff with respect to the early exercise threshold values for each of the initial threshold levels. The expectation term in Eq. (19), $E[\hat{J}_T(\theta_{10,000}]$, is the only one that need to be simulated, and it in fact corresponds to the price of a European call option. Thus we were able to apply the Black-Scholes formula, derived in Baxter and Rennie [1], in the gradient estimator Eq. (19) to compare with the performance of the stochastic algorithm using the simulated gradient. From the Table 1 solutions, we observe a close agreement between the analytical values of the Black-Scholes formula and the simulation values. Similar results are portrayed in Table 2 and Table 3. The measure of precision, given by the standard errors based on 10,000 samples, are shown in the tables with parentheses (in hundredths). This is an indication of how computationally efficient PA estimates are in Monte Carlo simulation. For D = 0.5, we have a maximum expected payoff value of 3.385 given by an exercise threshold value of approximately 55. Exercise threshold values above 55, as seen in Table 1, gave an expected payoff value close to the optimal. This trend is not borne in Table 2 and Table 3 with higher dividend amounts, because a lower cash dividend makes the behavior of the American option closer to a European-type option. Table 3, with the highest dividend amount, clearly shows this distinction. The expected option payoff values are seen to drop steadily as the exercise threshold values increase above the optimal value of approximately 54 with an expected payoff of 2.915.

Table 1:
D = 0.5, K = $S_0 = 50, r = 0.10,$
$\sigma = 0.3, (\tau_1, \tau_2) = (60, 30).$

Initial Level (s_n)	**Simulation Expected Payoff** (\$)	**Black-Scholes Payoff** (\$)	**Optimal Level** (s^*)
50	3.173 (0.256)	3.099	57.08
51	3.215 (0.204)	3.221	56.45
52	3.285 (0.217)	3.279	56.17
53	3.279 (0.246)	3.266	56.98
54	2.999 (0.213)	3.075	56.25
55	3.385 (0.194)	3.399	55.44
56	3.327 (0.224)	3.325	56.12
57	3.299 (0.331)	3.312	55.80
58	3.278 (0.291)	3.294	55.96
59	3.202 (0.341)	3.162	56.12

Table 2:
D = 1.0, K = $S_0 = 50, r = 0.10,$
$\sigma = 0.3, (\tau_1, \tau_2) = (60, 30).$

Initial Level (s_n)	**Simulation Expected Payoff** (\$)	**Black-Scholes Payoff** (\$)	**Optimal Level** (s^*)
50	2.941 (0.373)	2.918	56.94
51	2.817 (0.279)	3.044	56.19
52	2.985 (0.247)	2.992	56.22
53	3.237 (0.223)	3.249	55.33
54	3.306 (0.208)	3.294	55.05
55	3.159 (0.236)	3.184	55.64
56	3.093 (0.248)	3.105	56.07
57	3.005 (0.277)	3.019	56.11
58	2.897 (0.286)	2.994	57.07
59	2.642 (0.375)	2.699	57.23

Table 3:
D = 1.5, K = $S_0 = 50, r = 0.10$,
$\sigma = 0.3, (\tau_1, \tau_2) = (60, 30)$.

Initial Level (s_n)	**Simulation Expected Payoff** (\$)	**Black-Scholes Payoff** (\$)	**Optimal Level** (s^*)
50	2.798 (0.331)	2.603	55.99
51	2.841 (0.312)	2.883	55.01
52	2.915 (0.210)	3.097	54.11
53	2.891 (0.270)	2.978	54.55
54	2.867 (0.301)	2.814	55.01
55	2.670 (0.322)	2.671	55.14
56	2.505 (0.329)	2.561	55.27
57	2.495 (0.338)	2.510	55.65
58	2.433 (0.355)	2.464	55.72
59	2.374 (0.409)	2.383	55.99

7. CONCLUSION

Although Monte Carlo simulation is useful for pricing complex options markets, it could be time consuming and expensive depending on how it is implemented. In this paper, we investigated the use of gradient estimation techniques that were efficiently carried out in Monte Carlo simulation. Using perturbation analysis techniques we obtained both an estimate of the option value together with estimates of sensitivities of the option value to various parameters of the model. These sensitivity estimates were used in a stochastic approximation algorithm, called a Robbins-Monro-like algorithm, in order to maximize the expected return of the option payoff. Due to the generality of Monte Carlo simulation, these techniques can be applied to a growing subset of option pricing problems.

8. REFERENCES

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