A Newsvendor Model with Initial Inventory and Two Salvage Opportunities

Ali CHEAITOU
Euromed Management
Marseille, 13288, France

Christian VAN DELFT
HEC School of Management, Paris (GREGHEC)
Jouys-en-Josas, 78351, France

Zied JEMAI
Laboratoire Genie Industriel, Ecole Centrale Paris
Châténay-Malabry, 92295, France

and

Yves DALLERY
Laboratoire Genie Industriel, Ecole Centrale Paris
Châténay-Malabry, 92295, France

ABSTRACT

In this paper, we develop an extension of the newsvendor model with initial inventory. In addition to the usual quantity ordered at the beginning of the horizon and the usual quantity salvaged at the end of the horizon, we introduce a new decision variable: a salvage opportunity at the beginning of the horizon, which might be used in the case of high initial inventory level. We develop the expression of the optimal policy for this extended model, for a general demand distribution. The structure of this optimal policy is particular and is characterized by two threshold levels. Some managerial insights are given via numerical examples.

Keywords: Newsvendor model, initial inventory, lost sales, salvage opportunities, concave optimisation and threshold levels.

1. INTRODUCTION

The single period inventory model known as the newsvendor model is an important paradigm in operations research and operations management literature. It has had numerous important applications, as in style-goods products (fashion, apparel, toys, etc.) or in services management (booking on hotels, airlines, etc).

Many extensions have been proposed in order to include specific additional characteristics in the original newsvendor model. The literature concerning the newsvendor model is thus very large (for extensive literature reviews, see for example [1]). Generally speaking, a newsvendor model is characterized by three elements: the objective function, the demand characterization and different financial flow specifications. Most of the studies about the newsvendor model focus on the computation of the optimal order quantity that maximizes the expected profit (or minimizes the expected cost) [2]. Nevertheless, some other works consider other criteria, such as maximizing the probability of achieving a target profit [1]. The demand process can be considered exogenous [2] or price-sensitive [3]. Financial flows generally introduced in the newsvendor problem are the wholesale price, the selling price, the salvage value and the shortage penalty cost. Many extensions exist, such as a fixed ordering cost [4] or a dynamic selling price [5].

Other authors consider the newsvendor problem with a multiplicative neutral independent background risk in an expected utility framework [6]. Some studies treat the newsvendor model in a loss aversion or risk-averse framework [7].

In some extensions, other decision variables or parameters have been considered. For example, [8] have analyzed a newsvendor model with an initial inventory. In this extension, the decision maker observes, at the beginning of the selling season, the initial inventory level and fixes his decisions as a function of this initial inventory. These authors have shown that in this case the optimal order quantity can be deduced from the classical model (without initial inventory). [9] has considered a similar model in which the vendor, after observing the demand value, can carry out partial returns or additional orders in the limit of defined levels. [10] has studied the multi-product newsvendor problem with value-at-risk considerations.

In the present paper, we develop a new extension of the initial inventory newsvendor model in which a part of the initial inventory can be salvaged at the beginning of the selling season. As a matter of fact, when the initial inventory level is sufficiently high, it may be profitable to immediately salvage a part of this initial inventory to a parallel market, before the season. This is an extension of the classical model in which the unique salvage opportunity is placed at the end of the selling season.

In many practical situations, a potential interest exists for such a salvage opportunity before the selling season. For example, if a first quantity is ordered from the supplier a long time before the season, due to very long design/production/delivery lead-times, the demand distribution is not precisely known at the date of the order [11]. In this case, if the demand appears to be particularly low, it could be profitable to return a part of the received quantity to the supplier or sell it to a parallel market, with a return price which is lower than the order price. In this paper, we establish that the optimal policy corresponding to our model is a threshold based policy with two different thresholds: the first corresponds to the order-up-to-level policy of the classical model with initial inventory, and the second threshold corresponds to a
salvage-up-to-level policy, and is a result of the salvage opportunity at the beginning of the season. Between the two thresholds, the optimal policy consists of neither ordering, nor salvaging any quantity.

The remainder of this paper is structured as follows. In the following section, we introduce the model, describe the decision process and define the notation used in the paper, the objective function and the model assumptions. In Section 3, we show some of our model properties, we solve the model and exhibit the structure of the optimal policy as a function of the initial inventory level. In section 4 we give some managerial insights via numerical applications. The last section is dedicated to conclusions and presentation of new avenues of research.

2. THE MODEL PARAMETERS

A manager has to fill an inventory in order to face a stochastic demand. The ordering and selling processes are as depicted in Figure 1.

Before occurrence of the demand, an initial inventory is available. Without loss of generality and because it is more coherent with the main idea of the present paper, this inventory is assumed to be positive. Note however that a model with a negative initial inventory is assumed with the main idea of the present paper, this inventory is assumed able. Without loss of generality and because it is more coherent properties, we solve the model and exhibit the structure of the optimal policy as a function of the initial inventory level.

We also define the following parameters:

- **D**: the random demand, which is characterized by a continuous probability density function \( f(\cdot) : [0, \infty] \to \mathbb{R}^+ \) and by the cumulative distribution function \( F(\cdot) : [0, \infty] \to [0, 1] \);
- \( p \): the unit selling price during the season;
- \( s_b \): the unit salvage value for the quantity \( S_b \);
- \( c \): the unit order cost for the quantity \( Q \);
- \( s_e \): the unit salvage value of the quantity \( S_e \);
- \( b \): the unit shortage penalty cost.

As mentioned above, the objective function of the model consists of maximizing the total expected profit, denoted as \( \Pi(I_b, Q, S_b, S_e) \). This expected profit, with respect to the random variable \( D \), is explicitly given by

\[
\Pi(I_b, Q, S_b, S_e) = \int_{0}^{\infty} D f(D) dD + \int_{0}^{\infty} f(D) dD
\]

\[
- b \int_{0}^{\infty} \int_{I_b + Q - S_b}^{\infty} (D - I_b - Q + S_b) f(D) dD
\]

The different terms can be interpreted as follows: \( s_b S_b \), is the profit generated by salvage at the beginning of the season; \( cQ \), is the order purchase cost; \( s_e S_e \), is the profit generated by salvage at the end of the season; the fourth and fifth terms are the expected sales; the last term is the expected shortage penalty cost.

It is worth noting that equivalent models can be built with a cost minimization criterion ([1] and [12]). The decision variables have to satisfy the following constraints,

\[
0 \leq Q
\]

\[
0 \leq S_b \leq I_b
\]

\[
0 \leq S_e \leq I_e
\]

Some assumptions are necessary to guarantee the interest and the coherency of the model, as in the classical newsvendor model. These assumptions can be summarized in the following inequalities:

\[
0 < s_e < s_b < c < p
\]

Note that \( s_b \) and \( s_e \) can be negative, which corresponds to a situation where a cost is charged in order to dispose of the material.

3. THE MODEL

The Model Properties

In this section we consider the model described in Section 2 and we show the concavity of its expected objective function with respect to the decision variables, which permits to explore the structure of the optimal policy.

**Property 1** The objective function \( \Pi(I_b, Q, S_b, S_e) \), defined in Eq. (1) is a concave function with respect to \( Q, S_b \) and \( S_e \).

**Proof 1** The hessian of \( \Pi(I_b, Q, S_b, S_e) \) with respect to \( Q, S_b \) and \( S_e \) is given by

\[
\nabla^2 \Pi(I_b, Q, S_b, S_e) = -(b + p) f(I_b + Q - S_b) \begin{bmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]
From the model assumptions (Eq. (5)), for each vector \(V = (V_1, V_2, V_3) \in \mathbb{R}^3\) we find

\[
V^T \nabla^2 \Pi(I_b, Q, S_b, S_c)V = -(b + p)f(I_b + Q - S_b)(V_1 - V_2)^2 \leq 0, \quad (7)
\]

which proves that the matrix \( \nabla^2 \Pi(I_b, Q, S_b, S_c) \) is semi-definite negative. Consequently, the objective function \( \Pi(I_b, Q, S_b, S_c) \) is jointly concave with respect to \( Q, S_b \) and \( S_c \).

**Lemma 1** The optimal value of the decision variable \( S_c \) is given by

\[
S_c^* = \max(0; I_c) \quad (8)
\]

**Proof 2** It could be easily shown that the first partial derivative of the expected objective function \( \Pi(I_b, Q, S_b, S_c) \) with respect to \( S_c \) is given by

\[
\frac{\partial \Pi(I_b, Q, S_b, S_c)}{\partial S_c} = s_c. \quad (9)
\]

From Lemma 1, one concludes that the optimal value of \( S_c \) depends on \( I_c \). However, at the beginning of the selling season, \( I_c \) is a random variable. Hence, if \( S_c \) is substituted by its expected optimal value in Eq. (1), then the expected profit function for the model described in Figure 1 becomes

\[
\Pi(I_b, Q, S_b) = s_bS_b - cQ + \int_{I_b+Q-S_b}^{I_b+Q-S_b} (I_b + Q - S_b - D)f(D) dD + p \int_{0}^{I_b+Q-S_b} Df(D) dD + p(I_b + Q - S_b) \int_{I_b+Q-S_b}^{\infty} f(D) dD - b \int_{I_b+Q-S_b}^{\infty} (D - I_b - Q + S_b)f(D) dD. \quad (10)
\]

It is worth noting that this expected objective function depends only on \( I_b, Q \) and \( S_b \). Therefore \( Q^*(I_b) \) and \( S_b^*(I_b) \), the optimal values of the decision variables \( Q \) and \( S_b \), are the solution of the optimization problem

\[
(Q^*(I_b), S_b^*(I_b)) = \arg \{ \max_{0 \leq Q, 0 \leq S_b \leq I_b} \{ \Pi(I_b, Q, S_b) \} \} \quad (11)
\]

where \( \Pi(I_b, Q, S_b) \) is given in Eq. (10).

Since the objective function \( \Pi(I_b, Q, S_b) \) is concave with respect to the decision variables \( Q \) and \( S_b \), hence one could use the first order optimality criterion in order to characterize the optimal policy.

**Optimality Conditions for \( Q^* \)**

Consider the partial derivative of \( \Pi(I_b, Q, S_b) \) with respect to \( Q \)

\[
\frac{\partial \Pi(I_b, Q, S_b)}{\partial Q} = -c + b + p + (s_e - b - p)F(I_b + Q - S_b) \quad (12)
\]

For any given \( S_b \) value satisfying \( 0 \leq S_b \leq I_b \), the optimal ordering quantity \( Q^*(I_b) \) is a function of \( I_b - S_b \) that can be computed as the solution of the following optimization problem

\[
Q^*(I_b) = \arg \left\{ \max_{0 \leq Q} \{ \Pi(I_b, Q, S_b) \} \right\} \quad (13)
\]

By concavity of \( \Pi(I_b, Q, S_b) \) with respect to \( Q \), and for any given \( S_b \) value, the optimal solution \( Q^*(I_b) \) is given either by

\[
Q^*(I_b) = 0 \quad (14)
\]

if \( -c + b + p + (s_e - b - p)F(I_b - S_b) \leq 0 \), or by

\[
Q^*(I_b) = F^{-1} \left\{ \frac{b + p - c}{b + p - s_e} \right\} - I_b + S_b \geq 0 \quad (15)
\]

if \( -c + b + p + (s_e - b - p)F(I_b - S_b) \geq 0 \).

**Optimality Conditions for \( S_b^* \)**

The partial derivative of \( \Pi(I_b, Q, S_b) \) with respect to \( S_b \) is given by

\[
\frac{\partial \Pi(I_b, Q, S_b)}{\partial S_b} = s_b - b - p + (b + p - s_e)F(I_b + Q - S_b). \quad (16)
\]

For any given \( Q \) value satisfying \( 0 \leq Q \), the optimal ordering quantity \( S_b^*(I_b) \) is defined as the solution of the following optimization problem

\[
S_b^*(I_b) = \arg \left\{ \max_{0 \leq S_b \leq I_b} \{ \Pi(I_b, Q, S_b) \} \right\} \quad (17)
\]

By concavity of \( \Pi(I_b, Q, S_b) \) with respect to \( S_b \), and for any given \( Q \) value, the optimal solution \( S_b^*(I_b) \) is given either by

\[
S_b^*(I_b) = 0 \quad (18)
\]

if \( s_b - b - p + (b + p - s_e)F(I_b + Q) \leq 0 \), or by

\[
S_b^*(I_b) = F^{-1} \left\{ \frac{b + p - s_e}{b + p - s_e} \right\} - I_b - Q \geq 0 \quad (19)
\]

if \( s_b - b - p + (b + p - s_e)F(I_b + Q) \geq 0 \).

**Critical Threshold Levels**

From the above optimality conditions, two threshold levels appear to be of first importance in the optimal policy characterization,

\[
Y_1^* = F^{-1} \left\{ \frac{b + p - c}{b + p - s_e} \right\} \quad (20)
\]

with, from assumption (5), are related by:

\[
Y_1^* \leq Y_2^*. \quad (21)
\]

These threshold levels can be interpreted as values such as

\[
-c + b + p + (s_e - b - p)F(Y_1^*) = 0, \quad (22)
\]

and

\[
s_b - b - p + (b + p - s_e)F(Y_2^*) = 0. \quad (23)
\]
As the function $F(\cdot)$ is monotonously increasing, for any $I_b$ values such that $I_b < Y_1^*$ (resp. $I_b > Y_1^*$), one finds
\[ -c + b + p + (s_e - b - p)F(I_b) > 0 \]
(resp. $-c + b + p + (s_e - b - p)F(I_b) < 0$), \hspace{1cm} (24)
and for any $I_b$ values such that $I_b > Y_2^*$ (resp. $I_b < Y_2^*$), one finds
\[ s_b - b + p + (b + p - s_e)F(I_b) > 0 \]
(resp. $s_b - b + p + (b + p - s_e)F(I_b) < 0$). \hspace{1cm} (25)

Critical Threshold Levels and Structure of the Optimal Policy

We show below that the structure of the optimal policy is in fact fully characterized by these two threshold levels as depicted in Figure 2.

\[ Q^*(I_b) = Y_1^* - I_b \]
\[ S_b^*(I_b) = I_b - Y_2^* \]
\[ Q^*(I_b) = 0 \]
\[ S_b^*(I_b) = 0 \]
\[ Y_1^* \]
\[ Y_2^* \]
Initial inventory $I_b$

Figure 2: Structure of the optimal policy

**Lemma 2** For $Y_1^* \leq I_b \leq Y_2^*$, the optimal solution is given by
\[ Q^*(I_b) = S_b^*(I_b) = 0 \]. \hspace{1cm} (26)

**Proof 3** For $Y_1^* < I_b < Y_2^*$, one finds
\[ \frac{\partial \Pi(I_b, 0, 0)}{\partial Q} < 0 \] and \[ \frac{\partial \Pi(I_b, 0, 0)}{\partial S_b} < 0 \], \hspace{1cm} (27)
which induces, by concavity, that the solution $Q^*(I_b) = S_b^*(I_b) = 0$ is the optimum of the profit function for such $I_b$ values. If $Y_1^* = I_b$, one finds
\[ \frac{\partial \Pi(I_b, 0, 0)}{\partial Q} = 0 \] and \[ \frac{\partial \Pi(I_b, 0, 0)}{\partial S_b} < 0 \], \hspace{1cm} (28)
which leads to the same conclusion. If $Y_2^* = I_b$, one finds
\[ \frac{\partial \Pi(I_b, 0, 0)}{\partial Q} < 0 \] and \[ \frac{\partial \Pi(I_b, 0, 0)}{\partial S_b} = 0 \], \hspace{1cm} (29)
which leads to the same conclusion. \hspace{1cm} □

**Lemma 3** For $I_b \leq Y_1^*$, the optimal solution is given by
\[ Q^*(I_b) = Y_1^* - I_b \] and $S_b^*(I_b) = 0$. \hspace{1cm} (30)

**Proof 4** For $I_b \leq Y_1^*$, one finds that
\[ \frac{\partial \Pi(I_b, Y_1^* - I_b, 0)}{\partial Q} = 0 \] and \[ \frac{\partial \Pi(I_b, Y_1^* - I_b, 0)}{\partial S_b} < 0 \], \hspace{1cm} (31)
which induces, by concavity, that the solution $Q^*(I_b) = Y_1^* - I_b$ and $S_b^*(I_b) = 0$ is the optimum of the profit function for such $I_b$ values. \hspace{1cm} □

**Lemma 4** For $Y_2^* \leq I_b$, the optimal solution is given by
\[ Q^*(I_b) = 0 \] and $S_b^*(I_b) = I_b - Y_2^*$. \hspace{1cm} (32)

**Proof 5** For $Y_2^* \leq I_b$, one finds that
\[ \frac{\partial \Pi(I_b, 0, 0)}{\partial Q} < 0 \] and \[ \frac{\partial \Pi(I_b, 0, 0)}{\partial S_b} = 0 \], \hspace{1cm} (33)
which induces, by concavity, that the solution $Q^*(I_b) = 0$ and $S_b^*(I_b) = I_b - Y_2^*$ is the optimum of the profit function for such $I_b$ values. \hspace{1cm} □

As in the classical **Newsvendor** model, it follows from the previous derivations that the optimal policy does not explicitly depend on the pair $(p, b)$ but only on the sum $p + b$. In particular, a model with a unit selling price $p$ and a penalty cost $b > 0$ is equivalent to a model with a penalty cost $b' = 0$ and a unit selling price $p' = p + b$.

4. **Numerical Examples and Insights**

The fundamental properties of the considered model will be illustrated by some numerical examples. In a first example, we illustrate the structure of the optimal policy as a function of the initial inventory $I_b$. Then we exhibit, via a second numerical example, the impact of the demand variability on the structure of optimal policy. A third example illustrates the effect of $s_e$, the salvage value at the beginning of the horizon. In the last example, we compare the considered extended model with the classical **newsvendor** model with initial inventory, and we show the potential benefit associated with the initial salvage process.

For these numerical applications, we assume that the demand has a truncated-normal distribution, corresponding to a normal distributed demand, $D \sim N[\mu, \sigma]$ truncated at the zero value (we exclude negative demand values). Without loss of generality we also assume that the inventory shortage cost is zero, namely $b = 0$.

In the following figures, $Q^*(I_b)$ and $S_b^*(I_b)$ represent the optimal values of the decision variables, and $E[S_b^*(I_b)]$ is the expected optimal value of the decision variable $S_b(I_b)$, which is given by
\[ E[S_b^*(I_b)] = \int_0^{I_b + Q^*(I_b) - S_b^*(I_b)} (I_b + Q^*(I_b) - S_b^*(I_b)) f(D) dD. \hspace{1cm} (34) \]
This is to account for the fact that the variables $Q$ and $S_b$ are decided before the demand is known while the variable $S_e$ is decided after the demand is realized.
Optimal Policy

In this first example, we depict the behaviour of the optimal decision variables as a function of the initial inventory $I_b$. The numerical values for the parameters are the following: $\mu = 1000$, $\sigma = 400$, $p = 100$, $s_b = 30$, $c = 50$ and $s_e = 20$.

The two thresholds $Y_1^* = 1127$ and $Y_2^* = 1460$ have been represented in Figure 3. For $Y_1^* \leq I_b \leq Y_2^*$, one has $Q^*(I_b) = S_b^*(I_b) = 0$, while $E[S_e^*(I_b)]$ is increasing. For $I_b < Y_1^*$, $Q^*$ decreases linearly as a function of $I_b$, which corresponds to the order-up-to-level policy defined in Section 3. For $I_b > Y_2^*$, $S_e^*(I_b) > 0$ is a linear increasing function of $I_b$, which corresponds to the salvage-up-to-level policy defined in Section 3.

Variability Effect

In this extended model, the decision variables $Q$ and $S_b$ are fixed before demand occurrence and are thus, in one way or another, affected by demand variability. On the other hand, the decision variable $S_e$ is fixed once the demand is perfectly known. From an intuitive point of view, the more variable the demand the more profitable is postponement of the decisions.

Effect of Initial Salvage Value $s_b$

In order to show the consequence of the variability, we compare the example of Figure 3 with two other examples with higher (Figure 4) and lower (Figure 5) variability. All the numerical parameters are the same as in the first example, except demand variability. The demand standard deviations are respectively $\sigma = 600$ for the second example in Figure 4 and $\sigma = 200$ for the third example in Figure 5.

The thresholds are $Y_1^* = 1191$ and $Y_2^* = 1690$ for Figure 4 and $Y_1^* = 1064$ and $Y_2^* = 1230$ for Figure 5. It can be seen that the values of both thresholds increase with the standard deviation of the demand. The increase of $Y_1^*$ is in accordance with standard newsvendor results: with normally distributed demands, where for given costs, the order quantity increases linearly with the demand standard deviation. The same justification is valid for the increase of $Y_2^*$. This increase in the $Y_1^*$ and the $Y_2^*$ values is accompanied by an increase of the optimal $Q^*$ or the decrease of the $S_e^*$ for a given initial inventory value. The increase of the optimal $Q$ value permits the manager, for a given initial inventory, to stock a bigger quantity to face demand variability. The same is true for the decrease of the optimal $S_b$ value.

The increase in the $Y_1^*$ and $Y_2^*$ values is accompanied by an increase in the difference $Y_2^* - Y_1^*$. In the interval $[Y_1^*, Y_2^*]$, the value of $E[S_e^*(I_b)]$ increases with $I_b$. The fact that the interval width $Y_2^* - Y_1^*$ increases with demand standard deviation leads to an increase of $E[S_e^*(I_b)]$, which may be interpreted as a postponement of the decision until season’s end.

Figure 3: Optimal policy

![Figure 3: Optimal policy](image)

Figure 4: Optimal policy for high demand variability

![Figure 4: Optimal policy](image)

Figure 5: Optimal policy for low demand variability

![Figure 5: Optimal policy](image)
An increase of $s_b$ automatically induces a decrease of $Y^*_b$, which means that for a given value of the initial inventory $I_b$, the salvaged quantity $S^*_b(I_b)$ will increase. This increase will be accompanied by a decrease of the expected value of $S^*_b(I_b)$. This can be summarized as follows: the higher the salvage value of the parallel market, the higher the salvaged quantity at the end of the season $E[S^*_b(I_b)]$.

Numerical Comparison with the Classical Newsvendor Model with Initial Inventory

Our extended model introduces the additional variable $S_b$, which appears to be useful in presence of high initial inventory level. In order to illustrate the magnitude of the benefits potentially associated with $S_b$, we compare our model with the classical initial inventory newsvendor model, where $s_b = 0$. For the same numerical parameters values, we have measured the relative difference between the expected objective (profit) functions of the two models. We have considered three values of the salvage value $s_b$ for our model: the nominal value, $s_b = 30$; a high value, $s_b = 35$; a low value, $s_b = 25$. The comparison is shown in Figure 8.

Figure 8 shows that the benefits associated with the $S_b$ variable can be non-negligible for high values of $I_b$. Clearly, it is equal to zero for the $I_b$ values that are less than $Y^*_b$, where $S^*_b = 0$. Via Figure 8, one may conclude that:

- the difference, between the two expected optimal objective functions, is greater for high $s_b$ values. This increase corresponds logically to the fact that the $s_b$ term only appears in the objective function of the extended model and not in the newsvendor model.
- the threshold $Y^*_b$ decreases with $s_b$. For high $s_b$ value, the difference becomes positive.

This can be summarized as follows: the extended model is profitable for high $s_b$ values and/or high $I_b$ values.

5. SUMMARY AND CONCLUSIONS

This paper presents a new extension to the initial inventory newsvendor model in which a part of the initial inventory can be salvaged to a parallel market before demand occurrence. We have shown that in the case of a high initial inventory level, or a high initial salvage value $s_b$, this feature can be useful. The structure of the optimal policy is characterized by two threshold levels. Via numerical applications, we have illustrated the theoretical properties and given some managerial insight.

The extension of this model to a multi-periodic framework or to a model with pricing decisions is an ongoing research avenue.

References