Operation-Based Notation for Archimedean Graph

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ABSTRACT

We introduce three graph operations corresponding to polyhedral operations. By applying these operations, thirteen Archimedean graphs can be generated from Platonic graphs that are used as seed graphs.

Keyword: Archimedean graph, Polyhedral graph, Polyhedron notation, Graph operation.

1. INTRODUCTION

Archimedean graph is a simple planar graph isomorphic to the skeleton or wire-frame of the Archimedean solid. There are thirteen Archimedean solids, which are semi-regular polyhedra and the subset of uniform polyhedra. Seven of them can be formed by truncation of Platonic solids, and all of them can be formed by polyhedral operations defined as Conway polyhedron notation [1-2].

The author has recently developed an interactive modeling system of uniform polyhedra including Platonic solids, Archimedean solids and Kepler-Poinsot solids, based on graph drawing and simulated elasticity, mainly for educational purpose [3-5]. Obviously, it is possible to draw a polyhedral graph only with two graph operations: vertex addition and edge addition. However there is an individual difference, it might be time-consuming to input a graph with steady steps, especially in the case of Archimedean graph.

In this paper, three graph operations corresponding to polyhedral operations are introduced, and it is shown that every Archimedean graph can be generated using these operations and a Platonic graph as the seed graph.

2. ARCHIMEDEAN SOLIDS

Thirteen Archimedean solids are listed in Table 1, and illustrated in Figures 1-2. The symbols in the table stand for the vertex configurations. For example, \( A_{3,4} \) indicates that two regular triangles and two squares are gathered alternately on each vertex. Archimedean solids are surrounded by several sorts of congruent regular polygons, and their vertex figures are not regular but congruent polygons. The term vertex figure was introduced by H. Coxeter as the segment joining the mid-point of the two sides through a vertex [6].

Table 1. The list of Archimedean solids, where \( p, q, r \) are the number of vertices, edges, and faces, respectively.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name of polyhedron</th>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{3,4} )</td>
<td>Cuboctahedron</td>
<td>12</td>
<td>24</td>
<td>14</td>
</tr>
<tr>
<td>( A_{4,6,10} )</td>
<td>Great Rhombicosidodecahedron</td>
<td>120</td>
<td>180</td>
<td>62</td>
</tr>
<tr>
<td>( A_{4,6,8} )</td>
<td>Great Rhombicuboctahedron</td>
<td>48</td>
<td>72</td>
<td>26</td>
</tr>
<tr>
<td>( A_{3,5,5} )</td>
<td>Icosidodecahedron</td>
<td>30</td>
<td>60</td>
<td>32</td>
</tr>
<tr>
<td>( A_{3,4,5,4} )</td>
<td>Small Rhombicosidodecahedron</td>
<td>60</td>
<td>120</td>
<td>62</td>
</tr>
<tr>
<td>( A_{3,4} )</td>
<td>Small Rhombicuboctahedron</td>
<td>24</td>
<td>48</td>
<td>26</td>
</tr>
<tr>
<td>( A_{3,4} )</td>
<td>Snub Cube</td>
<td>24</td>
<td>60</td>
<td>38</td>
</tr>
<tr>
<td>( A_{3,5} )</td>
<td>Snub Dodecahedron</td>
<td>60</td>
<td>150</td>
<td>92</td>
</tr>
<tr>
<td>( A_{3,3,4} )</td>
<td>Truncated Cube</td>
<td>24</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>( A_{3,3,10} )</td>
<td>Truncated Dodecahedron</td>
<td>60</td>
<td>90</td>
<td>32</td>
</tr>
<tr>
<td>( A_{3,5,5} )</td>
<td>Truncated Icosahedron</td>
<td>60</td>
<td>90</td>
<td>32</td>
</tr>
<tr>
<td>( A_{4,3,4} )</td>
<td>Truncated Octahedron</td>
<td>24</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>( A_{3,3,6} )</td>
<td>Truncated Tetrahedron</td>
<td>12</td>
<td>18</td>
<td>8</td>
</tr>
</tbody>
</table>

Figure 1. Appearance of thirteen Archimedean solids (1/2).
3. GRAPH OPERATIONS

Thirteen Archimedean graphs isomorphic to the skeletons of Archimedean solids are depicted in Figure 3. A graph \( G = (V,E) \) is defined by the set of vertices \( V = \{v_0, \ldots, v_{p-1}\} \) and edges \( E = \{e_0, \ldots, e_{q-1}\} \). All graphs considered in this paper are 3-, 4-, or 5-regular polyhedral graphs, which are simple, planar and 3-connected graphs [7]. Detection of faces with \( n \) sides is equivalent to finding \( n \)-cycles. After selecting faces \( F = \{f_0, \ldots, f_{r-1}\} \), a polyhedral graph is redefined as \( \tilde{G} = (V,E,F) \) [3]. The set of faces is subdivided as follows,

\[
F = F_3 \cup F_4 \cup F_5 \cup F_6 \cup F_8 \cup F_{10}, \quad \forall i \neq j : F_i \cap F_j = \emptyset,
\]

where \( F_n \) denotes the set of faces with \( n \) sides.

We define three graph operations for \( \tilde{G} = (V,E,F) \) : diagonal addition, edge contraction, and vertex splitting. Examples are shown in Figures 4-6. A diagonal addition to a face in \( F_4 \) is an edge addition between a pair of non-adjacent vertices in a quadrangular face. An edge contraction is a graph contraction of an edge. A vertex splitting is defined conventionally as the reverse of edge contraction, but in this paper, we define a vertex splitting of \( v \) as the composition of the operations of subdivision of incident edges on \( v \), connecting the new vertices in a proper order, and deleting the vertex \( v \). Conventional vertex splitting is equivalent to the present vertex splitting followed by several edge contractions.

Figure 4. An example of diagonal addition operations.

Figure 5. An example of edge contraction operations.

Figure 6. An example of vertex splitting operations by the present definition in this paper.
4. OPERATION-BASED NOTATION OF ARCHIMEDEAN GRAPH

The symbol $\delta$ stands for applying diagonal addition to each quadrangular face of $F_4$ which is adjacent to two triangular faces of $F_3$. The operator $[m,n][n,m]$ stands for applying edge contraction to each edge incident on two faces of $mF$ and $nF$. The operator $\sigma$ stands for applying vertex splitting to every vertex.

The operator “truncate” in the Conway polyhedron notation [1-2] corresponds to the graph operation $\sigma$. The operator “ambo”, “expand” and “snub” are expressed by $a_{2,3}^{b,c}$, $a_{3,4}^{b,c}$, $a_{4,5}^{b,c}$, respectively. In the case of Archimedean graphs, equalities $lmn$ hold. We use following notations for Platonic graphs and Platonic solids: cube $P_3$, dodecahedron $P_5$, icosahedron $P_5$, octahedron $P_4$, and tetrahedron $P_3$.

As a consequence, every Archimedean graph can be expressed using one Platonic graph and graph operations as follows.

$$A_{3,6} = \sigma P_3,$$
$$A_{4,6} = \sigma P_3 = [4,8] \sigma [8,8] \sigma P_3,$$
$$A_{3,8} = \sigma P_3 = [4,6] \sigma [6,6] \sigma P_3,$$
$$A_{3,4} = [6,6] \sigma C_4 = [8,8] \sigma P_3,$$
$$A_{4,6,8} = [6,8] A_{4,6} = [8,6] \sigma [6,6] \sigma P_3 = [6,8] \sigma [8,8] \sigma P_3,$$
$$A_{3,4} = \delta A_{3,4} = [8,6] \sigma [6,6] \sigma P_3 = [6,8] \sigma [8,8] \sigma P_3,$$
$$A_{6,6} = \sigma P_3 = [4,10] \sigma [10,10] \sigma P_3,$$
$$A_{3,8} = \sigma P_3 = [4,6] \sigma [6,6] \sigma P_3.$$

5. ADDITION OF THE GRAPH OPERATIONS TO AN INTERACTIVE MODELING SYSTEM

An interactive modeling system has been developed by the author [3-5]. The coordinate of vertices are computed without the knowledge of metric information, but only with the structure of the isomorphic polyhedral graph. The system consists of three subsystems: graph input subsystem, wire-frame subsystem, and polygon subsystem. The overviews of the subsystems are described in the continuing subsections.

Figure 7 shows the relations of 5 Platonic graphs and 13 Archimedean graphs. There are five more possibilities of edge contraction operations to Archimedean graphs: $A_{3,4}^{b,c}$, $A_{3,5}^{b,c}$, $A_{3,4}^{b,c}$, $A_{4,3}^{b,c}$, and $A_{3,3}^{b,c}$, which lead to trivial graph with one isolated vertex.

Figure 7. Relations of 5 Platonic graphs and 13 Archimedean graphs induced by diagonal addition, edge contraction and vertex splitting.
5.1 Graph Input Subsystem
Figure 8(a) shows a screen shot of GUI of graph input subsystem. Drawing a planar graph isomorphic to polyhedron is the first step of polyhedron modeling. In the subsystem, vertex addition, vertex deletion, edge addition, and edge deletion are implemented as fundamental operations. Vertices can be moved attended with the incident edges. In addition, edge contraction and vertex splitting are introduced.

5.2 Wire-Frame Subsystem
Figure 8(b) shows a screen shot of GUI of wire-frame subsystem. After constructing a polyhedral graph, the next step is arranging vertices in 3-dimesional space. We define three binary relations between two vertices: adjacent, neighbor, and diameter. The relation adjacent corresponds to the length of an edge in a 3-dimensional space. The relation neighbor means that the length of path between two vertices is 2, and two vertices are neighborhood of another vertex. It corresponds to the shape of vertex figure in a 3-dimensional space. The relation diameter means that the length of path between two vertices is the diameter of the graph. It corresponds to the circum-sphere of polyhedron. Virtual elastic forces are assumed between vertices according to these three relations and Hooke’s law. Wire-frame polyhedron can be formed semi-automatically by controlling the natural length of virtual springs corresponding to the three types of binary relations.

5.3 Polygon Subsystem
Figure 8(c) shows a screen shot of GUI of polygon subsystem. After arranging vertices in 3-dimensional space, the last step is detecting faces, selecting faces, and rendering the solid [3]. In the case of Platonic solids, Archimedean solids, prisms, and anti-prisms, common routine is used. The faces of Kepler-Poinsot solids are detected by separate routine.

5.4 Addition of the graph operations
The graph operations of vertex splitting and edge contraction have been already implemented in the previous version of the graph input subsystem, but they are applied to individual vertex and edge respectively. In the present version, simultaneous vertex splitting of all the vertices is realized, and simultaneous edge contraction of all the edges between specified faces are also implemented. Such operations require 3-dimensional coordinate of each vertex and configuration of each face, therefore, they are available in the wire-frame subsystem and polygon subsystem. In these subsystems, vertex splitting and edge contraction are displayed as animation.

6. REFERENCES