Existence of Solutions for Linear Moment Problems

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ABSTRACT

The Cimmino algorithm is an iterative projection method for finding almost common points of measurable families of closed convex sets in a Hilbert space. It produces weak approximations of solutions for Fredholm equations of the first kind provided that solutions exist. We obtain an existence criterion for solutions of a linear moment of problem considered as Fredholm equation of the first kind and show that if this problems has a solution, then the Cimmino algorithm generate norm approximations of such solutions.

Keywords: almost common point, Cimmino type algorithm, Fredholm equation of the first kind, discrete linear moment problem, eigenvalue of a linear operator, Gram matrix.

1. PROBLEM STATEMENT AND MAIN RESULTS

Let X be a separable real or complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space and let $\{C_\omega\}_{\omega\in\Omega}$ be a family of nonempty closed convex subsets of the Hilbert space X such that the point-to-set mapping $\omega \to C_\omega$ is measurable. We say that the family $\{C_\omega\}_{\omega\in\Omega}$ is *square-integrable* if it has a square-integrable selector, that is, if there exists a measurable function $\xi : \Omega \to X$ such that $\|\xi(\cdot)\|^2$ is integrable and $\xi(\omega) \in C_\omega$ for μ-almost all $\omega \in \Omega$. In this case, for each $x \in X$, the function $\omega \to \|P_\omega x\|^2$, where P_ω denotes the metric projection onto the set C_ω, is integrable (see [5, Chapter 2]).

Consider the operator $\mathbf{P} : X \to X$ given by

$$\mathbf{P}x = \int_\Omega (P_\omega x) d\mu(\omega), \tag{1}$$

Definition 1 *The sequence $\left\{x^k\right\}_{k\in\mathbb{N}} \subset X$ such that*

$$x^0 \in X \quad \text{and} \quad x^{k+1} = \mathbf{P}x^k,\ \forall k \in \mathbb{N} \tag{2}$$

is called an orbit of the operator $\mathbf{P}$.

Roughly speaking, the investigation o fthe converhence of the iterations (2) is thew implementation of wel-known Cimmino algorithm [1], [2], [6], [7]

Let the function $\mathbf{g} : X \to [0, \infty)$ be given by

$$\mathbf{g}(x) = \int_\Omega \|P_\omega x - x\|^2 d\mu(\omega) \tag{3}$$

The function $\mathbf{g}$ is well-defined and finite everywhere.

Definition 2 *The set*

$$C := \{x \in X : x \in C_\omega,\ \mu\text{-a.e.}\}\,. \tag{4}$$

is called the set of μ-almost common points of the sets C_ω.

Clearly, this set is convex and closed.

Let $\{C_\omega\}_{\omega\in\Omega}$ *be a square-integrable family of nonempty, closed, convex subsets of X.*

The following theorems and corollaries have been proven in [4].

Theorem 3 *(A) The next five conditions are equivalent:*

(i) The set $Arg\,min\,g$ of (global) minimizers of the function g is nonempty;

(ii) The set $Fix\,P$ of fixed points of P is nonempty;

(iii) All orbits $\left\{x^k\right\}_{k\in\mathbb{N}}$ of P converge weakly to points in $Arg\,min\,g$;

(iv) All orbits $\left\{x^k\right\}_{k\in\mathbb{N}}$ of P are bounded;

(v) The operator P has a bounded orbit $\left\{x^k\right\}_{k\in\mathbb{N}}$.

(B) If $\left\{x^k\right\}_{k\in\mathbb{N}}$ is a bounded orbit of P, then the weak limit $z = w\text{-}\lim_{k\to\infty} x^k$ exists and the following conditions are satisfied:

(vii) The sequence $\left\{\mathbf{g}(x^k)\right\}_{k\in\mathbb{N}}$ converges to $g(z) = \min_{x\in X}\ g(x)$;

(viii) If $\min_{x\in X}\ g(x) = 0$, then $z \in C$,i.e. the set C is nonempty.

(C) If the set C of μ-almost common points of C_ω is nonempty, then the conditions above are also equivalent to the following one:

(vi) All the orbits $\{x^k\}_{k\in\mathbb{N}}$ of P converge weakly to points in C.

(D) If $Arg\, min\, g$ ($= Fix\, P$) has nonempty interior, then the any orbit $\{x^k\}_{k\in\mathbb{N}}$ of P converges strongly to a point in $Arg\, min\, g$. In particular, if C has nonempty interior, then any orbit $\{x^k\}_{k\in\mathbb{N}}$ of P converges strongly to a point in C.

Suppose that the functions $K : \Omega \to X$ and $b : \Omega \to R$ (or C) are measurable, $K(\omega) \neq 0$ for μ-almost all $\omega \in \Omega$, and that the function $\omega \to |b(\omega)| / \|\mathcal{K}(\omega)\|$ is μ-square integrable, that is, the next integral exists and

$$\int_\Omega \frac{|b(\omega)|^2}{\|\mathcal{K}(\omega)\|^2} d\mu(\omega) < \infty.$$

Consider the Fredholm equation of the first kind in the form

$$\langle K(\omega), x\rangle = b(\omega) \text{ a.e.on } \omega \in \Omega. \tag{5}$$

Obviously the set of solutions for equation (5) is the set of μ-*almost common points of the sets* C_ω,defined by

$$C_\omega = \{x \in X : \langle K(\omega), x\rangle = b(\omega)\}.$$

Corollary 4 *The following statements are true:*

(i) The point-to-set mapping $\omega \to C_\omega$ is measurable and has a μ-square integrable selector;

(ii) The operator $\mathbf{P} : X \to X$ given by

$$\mathbf{P}x = x + \int_\Omega \frac{b(\omega) - \langle \mathcal{K}(\omega), x\rangle}{\|\mathcal{K}(\omega)\|^2} \mathcal{K}(\omega) d\mu(\omega), \tag{6}$$

as well as the function

$$\mathbf{g}(x) = \int_\Omega \frac{|b(\omega) - \langle K(\omega), x\rangle|^2}{\|K(\omega)\|^2} d\mu(\omega), \tag{7}$$

are well-defined;

(iii) The equation (5) has solution if and only if there exists a bounded orbit $\{x^k\}_{k\in\mathbb{N}}$ of the operator $\mathbf{P}$ defined by (6) such that

$$\lim_{k\to\infty} \mathbf{g}(x^k) = 0. \tag{8}$$

(iv) If the equation (5) has solution, then any orbit $\{x^k\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ converges weakly to a solution of (5).

(v) If the function $\mathbf{g}$ is coercive (in the sense that $\lim_{\|x\|\to\infty} \mathbf{g}(x) = +\infty$), then all orbits of $\mathbf{P}$ converge weakly to fixed points of $\mathbf{P}$.

(vi) If $\mathbf{g}$ is coercive and there exists an orbit $\{x^k\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ such that (8) holds, then (5) has solutions and any orbit of $\mathbf{P}$ converges weakly to a solution of (5).

Corollary 4 shows that by establishing criteria for the coercivity of $\mathbf{g}$ we will obtain sufficient conditions for the weak convergence of the orbits of $\mathbf{P}$. To this end, we consider the linear, bounded, self-adjoint, positive semi-definite operator $M : X \to X$ given by

$$Mx = \int_\Omega \frac{\langle x, \mathcal{K}(\omega)\rangle}{\|\mathcal{K}(\omega)\|^2} \mathcal{K}(\omega) d\mu(\omega). \tag{9}$$

Note that the function M is well-defined because

$$\int_\Omega \left\| \frac{\langle x, \mathcal{K}(\omega)\rangle}{\|\mathcal{K}(\omega)\|^2} \mathcal{K}(\omega) \right\| d\mu(\omega) \leq \|x\|, \quad \forall x \in X.$$

We denote by σ_M the spectrum of M (see [11, p. 371]). This is a closed set of real numbers (see, for instance, [11, Theorems 9.2.1, 9.2.2, 9.2.3 and 10.4.2]) contained in the closed interval $[\alpha(M), \beta(M)]$ with

$$\alpha(M) = \inf_{\|x\|=1} \langle Mx, x\rangle \text{ and } \beta(M) = \sup_{\|x\|=1} \langle Mx, x\rangle, \tag{10}$$

having the properties that $\alpha(M), \beta(M) \in \sigma_M$ and

$$\|M\| = \beta(M) = \sup_{\|x\|=1} \langle Mx, x\rangle. \tag{11}$$

Since M is positive semi-definite, it follows from (10) that $\alpha(M) \geq 0$. Therefore, if M has an eigenvalue $\lambda \neq 0$, then we also have $0 < \lambda \leq \beta(M) = \|M\|$ showing that M is not identically zero.

Theorem 5 *If the linear operator M has $\alpha(M) > 0$, then the function $\mathbf{g}$ defined by (7) is coercive. Moreover, in these circumstances, if $\{x^k\}_{k\in\mathbb{N}}$ is an orbit of the operator $\mathbf{P}$ defined by (6), then $\{x^k\}_{k\in\mathbb{N}}$ is weakly convergent, its weak limit $z := w\text{-}\lim_{k\to\infty} x^k$ is a fixed point of $\mathbf{P}$ and a minimizer of $\mathbf{g}$, $\lim_{k\to\infty} \mathbf{g}(x^k) = \mathbf{g}(z)$, and one and only one of the following statements is true:*

(i) $\lim_{k\to\infty} \mathbf{g}(x^k) = 0$ in which case problem (5) has solutions and all orbits of $\mathbf{P}$ converge strongly to solutions of (5);

(ii) $\lim_{k\to\infty} \mathbf{g}(x^k) \neq 0$ in which case problem (5) has not solution.

2. DISCRETE LINEAR MOMENT PROBLEM

In this section we consider the set $\Omega := \mathbb{N}$. The set $\mathbb{N}$ is a complete probability space in which all subsets of $\mathbb{N}$ are measurable with the probabilistic measure defined by

$$\mu(A) = \sum_{j\in A} \mu_j, \tag{12}$$

where the sequence $\{\mu_j\}_{k\in\mathbb{N}}$ is such that $\sum_{j=0}^\infty \mu_j = 1$.

Discrete linear moment problems (DLMP for short) is formulated as follows:

Definition 6 *Let $\{K_j\}_{j\in\mathbb{N}} \subset X\backslash\{0\}$ and let $\{b_j\}_{j\in\mathbb{N}}$ be a scalar sequence. Find $x \in X$ such that*

$$\langle K_j, x\rangle = b_j, \quad \forall j \in \mathbb{N}. \tag{13}$$

It easily to see, that the problem (13) is a particular version of the Fredholm equation (5) with the functions $\mathcal{K}(j) = K_j$ and $b(j) = b_j$ for which all the conditions of Theorem 5 are satisfied. In this case we have

$$\mathbf{P}x = x + \sum_{j=0}^{\infty} (b_j - \langle K_j, x\rangle) K_j \frac{\mu_j}{\|K_j\|^2}, \tag{14}$$

$$\mathbf{g}(x) = \sum_{j=0}^{\infty} |b_j - \langle K_j, x\rangle|^2 \frac{\mu_j}{\|K_j\|^2}, \tag{15}$$

and

$$Mx = \sum_{j=0}^{\infty} \langle x, K_j\rangle K_j \frac{\mu_j}{\|K_j\|^2}, \tag{16}$$

where all the series are convergent.

Denote by G_n the Gram matrix of the first $n+1$ vectors $\sqrt{\mu_j}\frac{K_j}{\|K_j\|}$, $j = 0, 1, ..., n$ [8]. This matrix is positive Hermitian and, thus, it is real-valued and has real non-negative eigenvalues only (cf. [11, p. 469]) Denote by λ_n the minimal eigenvalue of the matrix G_n. It is well-known that

$$\lambda_n := \inf\left\{\bar{w}G_n w^T : w \in \mathbb{C}^{n+1}, \bar{w}\cdot w^T = 1\right\}, \tag{17}$$

so $\lambda_n \geq 0$, $n = 1, 2, ...,$and the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is nonincreasing, hence there exists $\lambda_* := \lim\limits_{n\to\infty} \lambda_n \geq 0$.

Theorem 7 *If*

$$\sum_{j=0}^{\infty} |b_j|^2 \frac{\mu_j}{\|K_j\|^2} < +\infty \tag{18}$$

and

$$\lim_{j\to\infty} \lambda_j > 0, \tag{19}$$

then the DLMP has a solution and any orbit $\left\{x^k\right\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ converges strongly to a solution of (13).

In accordance with Corollary 4 the implementation of the Cimmino algorithm to the discrete linear moment problems requires precise computation of the iterates x^k given by

$$x^{k+1} = x^k + \sum_{j=0}^{\infty} \left(b_j - \left\langle K_j, x^k\right\rangle\right) K_j \frac{\mu_j}{\|K_j\|^2}, \tag{20}$$

and this is rather difficult.. However it is shown that, if conditions 18 and 19 hold, then the Cimmino algorithm applied to the DLMP produces strong approximations of solutions to the DLMP by computing finite partial sums of the series 20.

It is obvious that if the sequences $\left\{|b_j|^2\right\}_{j\in\mathbb{N}}$ and $\left\{\|K_j\|^2\right\}_{j\in\mathbb{N}}$ are summable one can denote

$$\mu_k := \frac{\|K_k\|^2}{\sum\limits_{j=0}^{\infty} \|K_j\|^2}. \tag{21}$$

Hence the following corollary holds.

Corollary 8 *Let $\sum\limits_{j=1}^{\infty} \|K_j\|^2 < \infty$.If*

$$\sum_{j=0}^{\infty} |b_j|^2 < +\infty \tag{22}$$

and

$$\lim_{j\to\infty} \lambda_j > 0,$$

then the DLMP has a solution and any orbit $\left\{x^k\right\}_{k\in\mathbb{N}}$ of $\mathbf{P}$ converges strongly to a solution of (13).

Theorem 7 indicates a method of solving the DLMP by computing a large number of iterates x^k of an arbitrary orbit of $\mathbf{P}$. An intrinsic difficulty of this method is that it requires precise computation of the iterates x^k given by the rule

$$x^{k+1} = x^k + \sum_{j=0}^{\infty} \mu_j \frac{b_j - \left\langle K_j, x^k\right\rangle}{\|K_j\|^2} K_j, \tag{23}$$

Obviously, effectively computing the infinite sum occurring in (23) is rather difficult. This leads to the question whether Theorem 7 remains true if one replaces the iterates x^k given by (23) by "inexact" iterates y^k of the form

$$y^{k+1} = y^k + \sum_{j=0}^{n(k)} \mu_j \frac{b_j - \left\langle K_j, y^k\right\rangle}{\|K_j\|^2} K_j, \tag{24}$$

where, for each $k \in \mathbb{N}$, the nonnegative integer $n(k)$ is sufficiently large. The following result shows that this is indeed the case when the sequence $\left\{\frac{|b_j|}{Kj}\right\}_{j\in\mathbb{N}}$ has a known positive upper bound.

Corollary 9 *Let conditions of Theorem 7 hold, and let γ be a positive upper bound of the sequence $\{|b_j| / \|K_j\|\}_{j\in\mathbb{N}}$. If, for a summable sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ of positive real numbers, and for each $k \in \mathbb{N}$, the number $n(k)$ is chosen such that*

$$1 - \sum_{j=0}^{n(k)} \mu_j \leq \frac{\varepsilon_k}{\gamma + \|y^k\|}, \tag{25}$$

then any sequence $\left\{y^k\right\}_{k\in\mathbb{N}}$ given by (24) converges strongly to a solution of the discrete linear moment problem (13).

3. CONCLUSION AND APPLICATIONS

Conditions for the existence of solution for linear moment problem have been established and the iterative algorithm approximating a solution has been constracted.The moment problem considered in this paper has been generated by various problems of geometry, physics, mechanics [16]. One of important applications of linear moment problem in mathematics are the interpolation theory and the theory of Dirichle series [9],[12], optimal control theory [3], [10], controllability and observability theory for distributed systems [13], [14], [15].

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