Solution of the System Structure Reconstruction Problem Based on Generalization of Tellegen's Principle

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ABSTRACT

An extraordinary generality, conceptual simplicity and practical usefulness of the Tellegen's theorem is well known in the field of electrical engineering [1]. It is one of few general theoretical results that apply in non-linear and time-varying situations, too. For standard linear electrical network models with constant parameters many classical results of electrical circuits theory can be derived as direct consequences of it.
In the paper a more general class of abstract strictly causal system representations is addressed. A new problem, that of the abstract state space system representation structure reconstruction has been formulated in [3], and partially solved in [3] and [4]. In this paper a new approach based on a generalized form of the classical Tellegen's principle, providing an equivalence class of physically as well as mathematically correct solutions is developed and some well-known, as well as new results are shown to be straightforward consequences of the derived structure. Some connections of dissipativity, conservativity, state and parameter minimality, instability and chaos with system representation structures are investigated from this point of view. Analytical results are illustrated by a number of typical examples and visualized by simulations.
Keywords: physical and mathematical correctness, signal power balance principle, system energy additivity, equivalence relation, state and parametric minimality

1. INTRODUCTION

In many real-world situations some natural concepts, such as causality principle and different forms of conservation laws, have generally been recognized as system properties of crucial practical importance. In many specific fields of science and engineering such concepts have frequently been used as fundamental tools of system modelling and analysis. On the other hand there are many fields and/or real situations in which the laws of nature are not known or are not expressed in proper mathematically exact form. In such cases the main source of information on which the system representations relay is an identification procedure or an parameter estimation technique using experimentally gained data only [4], [5], [6]. It is standard way to classify the identification methods as non-parametric if no prior information about the system structure is assumed.
If real physical structure of the system under investigation would be explicitly known the so called parametric methods of identification could be used, and consequently more adequate modelling results should be expected [7]. Unfortunately, any reliable explicit knowledge of the physical system structure is more an exception as a rule.
One frequently used approach trying to attack the unknown structure problem is to choose the model structure ad hoc using some heuristic arguments, and verify whether the obtained quantitative results are not in contradiction with obvious qualitative expectations concerning the real system behavior and/or with results of additional experiments.
The main aim of the contribution is to put the fundamental system-theoretic question of the real internal system structure reconstructability and to investigate some possibilities of its systematic solution. The proposed approach gets out from the hypothesis that any physically correct system representation must not be in contradiction not only with a set of measured data but also with a form of *an energy-like conservation principle*. Hence introducing the additional requirement of a proper defined abstract form of energy conservation principle as an attribute of any causal system representation seems to be the most natural and appropriate way to do it.

2. STRUCTURE RECONSTRUCTION PROBLEM

At first, we introduce the natural concepts of physically correct and incorrect internal system structures using a few simple examples for motivation.

Example 1: Let us imagine that the following *n*-th order transfer function representation for $n = 3$

$$G(s) = \frac{y(s)}{u(s)} = \frac{b_1}{s^3 + a_1 s^2 + a_2 s + a_3} \qquad (1)$$

is given as a result of some identification procedure using some real data measured on a real system with *unknown internal structure*. The so called *realization problem* [5], is to find a triplet of matrices $\{A, B, C\}$ in such a way that it holds:

$$G(s) = C.\left[s\,I - A\right]^{-1}.B \qquad (2)$$

It is well known that the solution of this problem is *not unique*, because the specific structure of the matrices above *depends on the choice of the so called state variables* $x_1, x_2, \ldots x_n$. If no additional information about actual internal structure of the given real physical system is known, then many different choices are equivalent from the input-output relation (1), (2) point of view, and any of them can be considered *as equally mathematically correct* if a set of *state equivalence conditions* is satisfied.

One *natural* way how to *determine* an *internal structure* for the given system description can be motivated by converting G(s) *back into the time domain* and write:

$$y^{(n)}(t) + S(t) = w(t), \quad w(t) = B.u(t) \qquad (3)$$

where the *structure function* S defined as *scalar product* of a *parameter vector* and a *state vector representation:*

$$S = \langle x(t), \Theta \rangle \Leftrightarrow S(t) = \sum_{k=1}^{n} \Theta_k . x_k(t) \qquad (4)$$

describes the relation between state and structure of any system representation. Now, because the *parameter vector* Θ *is specified* by $G(s)$, *the most natural choice* of *state vector components* $x_i(t)$ follows from:

$$S = a_1 x_1 + a_2 x_2 + a_3 x_3 \qquad (5)$$

Thus *the resulting internal structure* of the chosen system representation is determined by *the matrices A, B, C* as follows:

$$B \triangleq \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -a_1, & -a_2, & -a_3 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \end{bmatrix}, \; C = [0, 0, 1], \quad (6)$$

and can be visualized equivalently by the oriented signal-flow graph G as shown at the *Fig.1.*

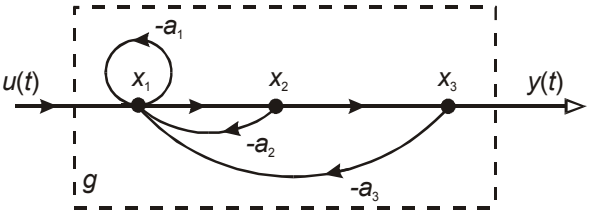


Fig.1. Internal structure induced by the state x(t)

It is easy to verify that the resulting system representation structure is mathematically correct for any real values of the parameters, but for the system order $n = 3$, or greater, such structures *can not be accepted as physically correct* in sense of input-state-output signal energy transfer if both the *signal power balance principle* and *signal energy additivity* are required to hold *simultaneously.*

To explain the situation, assume that the parameter values are such that the given system is *dissipative* and the *output signal power* P_o is defined by

$$P_o(t) = \|y(t)\|^2 = \langle C^T C\, x(t), x(t) \rangle = x_3^2(t) \qquad (7)$$

then *for any past- and for zero future input* the *signal power balance relation reads* [4]

$$\forall x(t_0), \; \forall t: \qquad \frac{dE_o}{dt}\left[x(t)\right] = -P_o(t) \qquad (8)$$

and is satisfied if and only if the energy $E_o(.)$ is given by:

$$E_o = x_3^2 + \frac{\Delta_1}{\Delta_2}\left(x_2 + \Delta_1 x_3\right)^2 + \frac{\Delta_1^2}{\Delta_3}\left(x_1 + \Delta_1 x_2 + \frac{\Delta_2}{\Delta_1} x_3\right)^2 \qquad (9)$$

$$\Delta_1 = a_1, \;\; \Delta_2 = a_1 a_2 - a_3, \;\; \Delta_3 = a_3(a_1 a_2 - a_3) \qquad (10)$$

that obviously *does not obey the additivity requirement*.

Example 2: Consider the same system (1), (3), but with a different internal structure. Now we choose the state variables in such a way that *the signal energy additivity*

$$E(\bar{x}) = \delta_1 E_1(\bar{x}_1) + \delta_2 E_2(\bar{x}_2) + \delta_3 E_3(\bar{x}_3) \qquad (11)$$

as well as the energy balance relation holds. It is easy to see that if $\bar{x}_1$ is chosen as the measured output signal then *a set of the state variables exists for which an equivalence class of physically correct structures can be found*. Such a *choice of the measured output power* induces the *physically correct internal structure* shown at the *Fig.2.*

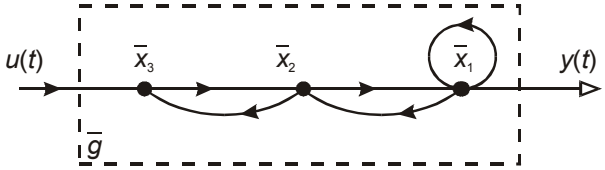


Fig.2. Internal structure induced by the output power

It is easy to see, that *the resulting chain structure, if realized* for example *by means of passive electrical components, is compatible* with the *actual internal structure* of an electrical network described by *matrices*

$$A = \begin{bmatrix} -\frac{1}{(R_1+R_2)C_1}, & \frac{1}{(R_1+R_2)C_1}, & \frac{1}{C_1} \\ \frac{1}{(R_1+R_2)C_2}, & -\frac{1}{(R_1+R_2)C_2}, & 0 \\ -\frac{1}{L}, & 0, & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{C_2} \\ 0 \end{bmatrix} \qquad (12)$$

and for $y(t) = i(t)$, $u(t) = \mathrm{I}(t)$ is shown at the *Fig.3.*

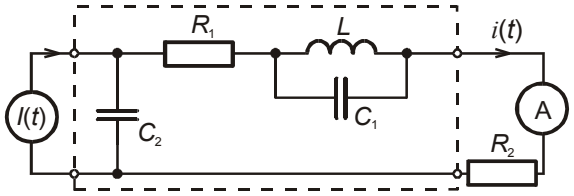


Fig.3. An example of physically correct structure for n=3.

The examples above can be seen as a motivation of the internal structure reconstructability concept and the *problem of system structure reconstruction* as follows:

Definition 1: *(Physical correctness)*

Assume, that an external representation

$$y^{(n)}(t) - w(t) = F\left[y, \dot{y}, \ldots, y^{(n-1)}, \Theta_1, \ldots, \Theta_n\right] \qquad (13)$$

of a strictly causal system *S* is known. For any given parametrization Θ_k *find a class of equivalent structure function*

representations $\overline{S}$ in such a way that any internal representation induced by

$$y^{(n)}(t) + \overline{S}(\overline{x},\overline{\Theta}) = w(t) \quad (14)$$

will *not be in contradiction with a form of signal power balance principle and* corresponding energy function will have the *additivity property*. Any such input-state-output representation will be called *physically correct*.

Definition 2: *(Structure reconstruction problem)*
Given an abstract mathematically correct system representation, then *the internal structure reconstruction problem (ISRP)* consists in finding such a specification of state and parameter vectors that physical correctness follows.
In the well developed field of electrical networks the well known Tellegen's theorem [1], has proven to be one of the most fruitfull results in this direction.

3. CLASSICAL TELLEGEN'S THEOREM AND CORRECTNESS OF ELECTRICAL CIRCUITS

In this section some basic ideas concerning mathematical and physical correctness of linear electrical networks are briefly summarized.
As the circuit theory postulates validity of Kirchhoff's laws (further KLs), we will strictly demand that the circuit be not in contradiction with the KLs.

Definition 3: *(Correctness of a circuit)*
A circuit satisfying both KLs is called (physically) correct.
A well known graph-theoretic result [2], gives the following criterion of the electrical circuit correctness.

Theorem 1: *(Criterion of correctness for a circuit)*
A circuit is correct if there is no loop coinciding with the voltage branches (i.e. $L \cap \beta_u = \varnothing$) and, at the same time, no set cut coinciding with the current branches (i.e. $C \cap \beta_i = \varnothing$).

It has been shown in [3] that a different approach based on more general system-theoretic concepts can be developed. The main results of this alternative approach are in fact based on the simple idea that the concept of correctness expresses two different requirements:

1. The property of mathematical correctness - elimination of all redundant elements of the mathematical description, i.e. the state- and parameter minimality of system representation,
2. The property of physical correctness - expressing the fact that some form of energy conservation principle has to hold for any physically correct representation. With the motivation above the following definition will be proposed.

Definition 4: *(Correctness of system representation)*
A network or a *system representation* will be called *correct* if it possesses two properties: the mathematical correctness and the physical correctness.
It is well known [1], that Kirchhof's laws are closely related to the classical Tellegen's theorem (CTT). Especially the fact that independently of the knowledge of system components the validity of energy conservation law can be proven as a direct consequence of CTT is of crucial importance for system structure determination. From the physical correctness problem point of view it is natural to ask if the classical formulation of CTT in terms of currents and voltages could be generalized in such a way that these electrical variables will be replaced by some *abstract variables without any a'priori defined physical meaning.*
Roughly speaking, the Tellegen's theorem asserts that Kirchhoff's laws are sufficient for energy conservation in an electrical network. The CTT is extremely general tool of system analysis and synthesis in electrical network theory; it is known to be valid for any lumped electrical network that contains any elements, linear and non-linear, passive and active, time-varying and/or time-invariant. This generality follows from the fact that Tellegen's theorem depends only on the two Kirchhof's laws.
In order to explain the essential features of CTT, consider an arbitrary connected electrical network of b components and choose associated reference directions for the branch voltages $v_k(t)$ and branch currents $i_k(t)$. In such notation the instantaneous value of the power $P_k(t)$ delivered at time t by the network to the branch k is given by the product:

$$P_k(t) = v_k(t)\ i_k(t) \quad (15)$$

Next, let us disregard the specific nature of the network components and represent the given network structure by an oriented graph G with n vertices and b branches. The Tellegen's theorem asserts that the total network power $P(t)$ satisfies the "power balance" relation:

$$P(t) = \sum_{k=1}^{b} P_k(t) = 0 \quad (16)$$

The only requirement on the branch voltages v_k is that they satisfy all the constraints imposed by the Kirchhoff's voltage (KVL); similarly the branch currents i_k have to satisfy all the constraints imposed by Kirchhoff's current law (KCL). The *specific nature of the network elements*, or, in fact, whether there are any elements that would have these i_k and v_k as branch variables, is *absolutely irrelevant* as far as the truth of CTT is concerned. Let the set of Kirchhoff law constraints be given by

$$A\,i = 0 \quad (17)$$

$$B\,v = 0 \quad (18)$$

where A is an *node incidence matrix*, B is an appropriate *loop incidence matrix*, and vectors i and v are defined

$$i = \left[i_1, i_2, ..., i_n\right]^T \quad (19)$$

$$v = \left[v_1, v_2, ... v_n\right]^T \quad (20)$$

Consider the vectors of network component currents and voltages to be elements of an b-dimensional Euclidean vector space E_b and define the inner product operation

$$\langle i, v \rangle = \sum_{k=1}^{b} i_k\, v_k \quad (21)$$

Let J be the set of all vectors i such that $i \in J$ if and only if i satisfies equation (17). Let V be the set of all vectors v such that $v \in V$ if and only if v satisfies equation (18).
The important principle known in the field of electrical network theory as the Tellegen's theorem [1] follows:

Theorem 2: *(Classical Tellegen's Theorem)*
If $i \in J$ and $v \in V$ then it holds

$$\forall t: \quad \langle i(t), v(t) \rangle = 0 \quad (22)$$

That is to say J and V are *orthogonal subspaces* of the Euclidean space E_b. Furthermore J and V together *span* E_b.

Remark 1: It is worthwhile to notice the close relation of physical correctness to the Tellegen's principle.

Remark 2: It is of crucial importance to realize that the branch voltages $v_1, v_2, \ldots, v_b$ are picked *arbitrarily* subject only to the KVL constraints. Similarly, the branch currents $i_1, i_2, \ldots, i_b$ are picked *arbitrarily* subject only the KCL. It means that if different sets $v_1, v_2, \ldots, v_b$, and $i_1, i_2, \ldots, i_b$ of arbitrarily selected branch voltages and branch currents satisfying the same KVL constraints and the same KCL constraints will be considered then we may apply the Eqn.(22) to the *new sets of variables* and obtain

$$\forall t: \quad \langle \overline{i}(t), \overline{v}(t) \rangle = 0 \quad (23)$$

Remark 3: The last condition will be used later as a motivation for introducing a *group of system equivalence transformations* on which a generalization of Tellegen's principle is based.

Remark 4: Using the abstract input-state-output causal system representations philosophy, it follows that an abstract generalized form of the Tellegen's relation has to hold, that *relates external variables*, input and output signals of a real system S to a *set of abstract internal variables* $x_i(t)$ *representing the state* of a proper chosen equivalence class of system representations in such a way that an abstract form of the *energy conservation principle holds structurally*, i.e. independently of the choice of system parameterization.

This is the key idea of the proposed approach to system structure reconstruction problem, as well as to nonlinear minimality, dissipativity, conservativity, instability and deterministic chaos, based on a generalization of classical Tellegen's principle.

4. GENERALIZED TELLEGEN'S PRINCIPLE - A NEW FORM OF ENERGY CONSERVATION LAW

From the system representation correctness point of view, as discussed above, it is clear that two different issues have to be distinguished: the so called *mathematical correctness* which seems to be equivalent to the state-and parameter minimality property of a causal system representation, and the *physical correctness* which is closely related to some form of signal power balance relation. Thus it is natural to require that besides of causality principle every physically correct system representation has to satisfy some form of the signal energy conservation law.

The main purpose of this section is to analyze the possibility of *generalizig the idea of Tellegen's principle* as presented above for more general system representations in which *the physical meaning of all the internal system variables is not a'priori known.*

Certainly, any realizable system has to fulfill some *causality* and *energy conservation* requirements. Recall that *existence of an abstract state space representation is necessary and sufficient* for a system to be *causal*. On the other hand causality does not imply energy conservation. In the field of electrical engineering *Kirchhoff's laws are necessary and sufficient for physical correctness of* any electrical network from energy conservation point of view. Tellegen's theorem, which is known to be one of the most powerful tools *of system analysis and synthesis* in electrical network theory, can be seen as a very elegant abstract form of *energy conservation principle* for a class *physically correct* system state space representations, in which voltages and currents have been chosen as state variables.

Let us briefly summarize the *essential features* of the Tellegen's theorem 2. Assume that an *arbitrary connected electrical network* of *b* components is given. Let us *disregard* the specific nature of the *network components* and represent the *network structure* by an *oriented graph* with *n vertices and b branches*. Let the *set of Kirchhoff law constraints* be given in a form (17), (18).

It is obvious fact, following directly from the *definition of inner product*, that relation (22) *is just a form of constant energy statement* for a class of representations in which *elements* of a *set* of *voltages and currents* have been *chosen* as *state variables*, as well as components of a *gradient vector of a scalar field in the state space.*

Let $\Re\{S\}$ is a *continuous-time* time-invariant strictly causal nonlinear system state space representation given by:

$$\Re\{S\}: \quad \dot{x}(t) = f[x(t)] + B\,u(t),\ x(t_0) = x^0, \qquad y(t) = C\,x(t), \tag{24}$$

The *arbitrariness* in the choice of state coordinates motivates introducing a *group of state- and feedback- transformations* on which the *generalization of classical Tellegen's principle* has been proposed in [4].

$$\exists\varphi,\ \exists T,\ T^{-1}: \ \bar{x} = T(x),\ \ \bar{u} = \varphi(u,\bar{x}): \quad \left\langle f, (grad\ \ E)^T \right\rangle = 0 \ \Leftrightarrow\ \left\langle \dot{\bar{x}}, \bar{x} \right\rangle = 0 \quad \Leftrightarrow \forall t: \ \bar{E}[\bar{x}(t)] = E[x(t)] \tag{25}$$

For a class of discrete-time finite dimensional *internal system representations* $\Re\{S\}$ given by

$$x(k+1) = f\left[x(k)\right] + w(k), \qquad w(k) = B\,u(k), \quad y(k) = C\,x(k) \tag{26}$$

Similarly as in the case of continuous-time systems, a new *discrete-time generalization of Tellegen's principle* has been introduced in [3]. If any input *u(k)* and any state value *x(k)* will be chosen then the next state value *x(k+1)* is given, and the *state difference vector Δx(k)* can be defined as

$$\Delta x(k) = x(k+1) - x(k) \equiv \Delta x_k, \quad k \in \{0,1,2,..\} \tag{27}$$

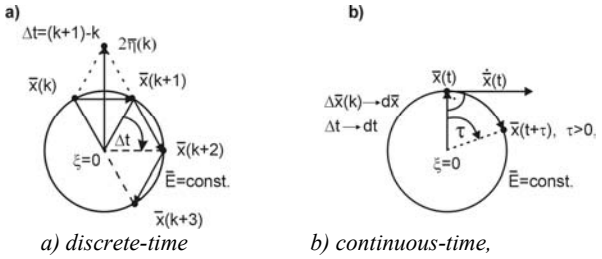
together with a row "*gradient vector" η(k)* defined by:

$$\eta(k) = \frac{1}{2}[x(k+1) + x(k)]^T \equiv \eta_k, \quad k \in \{\ 0,1,2,\dots\ \} \tag{28}$$

Interpretation of the vector η_k as a natural *discrete-time energy function gradient vector* is obvious, and the *discrete-time generalization of Tellegen's principle* is then given by the *inner product*:

$$\forall t \equiv k, k \in \{0,1,2,\dots\}: \ \left\langle \Delta x_k \ \ , \ \ \eta_k^{\ T} \right\rangle = 0 \tag{29}$$

For deeper understanding a *geometric interpretation* of the *generalized Tellegen's principle* is visualized at the *Fig.4.* with continuous-time version as a limit of the discrete-time case.



a) discrete-time *b) continuous-time,*

Fig.4. Geometric interpretation of the generalized Tellegen's principle (for n=2)

5. CONSERVATIVITY AND DISSIPATIVITY

Let us consider the class of continuous-time nonlinear time-varying strictly causal systems given by the state space representation

$$\Re\{S\}: \quad \dot{x}(t) = f[t; x(t), u(t)] \tag{30}$$

$$y(t) = h[t; x(t)] \tag{31}$$

with t as continuous time variable,

$x_1, x_2, \dots, x_n$ as the state space coordinates,

$\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n$ as coordinates of the state velocity,

$u_1, u_2, \dots, u_r$ as the input signals,

and with $y_1, y_2, \dots, y_p$ as the observed output signals.

Recall that according to Liouville's theorem of vector analysis, *dissipative systems* have the important property that any volume of the state space strictly decreases under the action of the system flow. For nonlinear system representations $\Re\{S\}$ with the state velocity given by a nonlinear vector field f the property of *dissipativity* is defined by using the *operation of divergence* as follows [4].

Definition 5: *(Dissipativity of a vector field)*

The representation $\Re\{S\}$ with the state velocity vector field f is *dissipative* if it holds

$$\operatorname{div} f(x) = \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i} < 0 \qquad (32)$$

Let us now define a *constituent set* of finite number of *non-interacting elementary subsystems*

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, u_1), & y_1 &= h_1(t, x_1) \\ \dot{x}_2 &= f_2(t, x_2, u_2,), & y_2, &= h_2(t, x_2) \\ &\cdots\cdots\cdots\cdots & &\cdots\cdots\cdots \\ \dot{x}_n &= f_n(t, x_n, u_n), & y_n &= h_n(t, x_n) \end{aligned} \qquad (33)$$

It follows that the *constituent set* (33) is *dissipative* if *at least one* of the elementary subsystems is dissipative.

Remark 5: It is easy to deduce that the constituent set of non-interacting subsystems with zero input and with unique equilibrium state is *locally asymptotic stable iff each of the elementary subsystems is dissipative.* It means that in general *dissipativity is necessary but not sufficient for asymptotic stability.*

Remark 6: Nonlinear systems having at an equilibrium state a dissipative approximate linearization are *locally dissipative* in a neighborhood of this equilibrium state, but need not to be *globally dissipative*, i.e. their *region of dissipation* need not be the whole state space.

Remark 7: Recall that systems with

$$\operatorname{div} f(x) = 0 \qquad (34)$$

preserve volume along state trajectories; such systems are usually referred to as *conservative.*

Notice that this concept of conservativity is *not always compatible* with the classical meaning of the term *conservative as energy preserving* (also called Hamiltonian).

Remark 8: Notice that a *linear time invariant system*

$$\begin{aligned} \Re\{S\}: \ \dot{x}(t) &= Ax(t) + Bu(t), \ x(t_0) = x^0, \\ y(t) &= Cx(t), \end{aligned} \qquad (35)$$

is dissipative if and only if its *matrix A has negative trace*, i.e. iff it holds

$$Tr\, A < 0 \qquad (36)$$

Thus an *asymptotically stable* linear system is *always dissipative,* while the *converse is not true in general*.

6. STATE MINIMALITY AND STATE EQUIVALENCE

Let's consider continuous-time linear time-varying strictly causal state space system representation $\Re\{S\}$:

$$\Re\{S\}: \ dx(t)/dt = A(t)x(t) + B(t)u(t), \ \dim x(.) = n, \dim u(.) = r \qquad (37)$$

$$y(t) = C(t)\, x(t), \ \dim y(.) = p, \qquad (38)$$

where $1 \le r \le n$, $1 \le p \le n$, and the matrices $A(.)$, $B(.)$, $C(.)$ are supposed to be known. Assume that the given system S has the asymptotic stability property and that its representation $\Re\{S\}$ has the minimal order n, i.e. it is controlable and observable. In such a case the controlability and observability Grammian matrices $W_c(t)$ and $W_o(t)$ exist, are symmetric and positive definite, and satisfy the Lyapunov-like equations:

$$A(t)\, W_c(t) + W_c(t)\, A^T(t) + dW_c(t)/dt = -B(t)\, B^T(t) \qquad (39)$$

$$A^T(t)\, W_o(t) + W_o(t)\, A(t) + dW_o(t)/dt = -C^T(t)\, C(t) \qquad (40)$$

Any such representation induces an equivalence class of minimal, controllable, observable and asymptotic stable state equivalent system representations given by the *conditions of state equivalence* in the form

$$\bar{A}(t) = [T(t)\, A(t) + dT(t)/dt\,]\, T^{-1}(t), \qquad (41)$$

$$\bar{B}(t) = T(t)\ B(t)\,, \qquad (42)$$

$$\bar{C}(t) = C(t)\, T^{-1}(t), \qquad (43)$$

generated by the group of linear time-varying state equivalence transformations:

$$\bar{x}(t) = T(t)\, x(t), \quad x(t) = T^{-1}(t)\, \bar{x}(t) \qquad (44)$$

It is well known fact that in the linear time-invariant case the Grammian matrices $W_c(t)$ and $W_o(t)$ are closely related to the input and output signal energy functions, respectively [2], [4].

In previous part it has been shown that the state minimality property of any linear strictly causal system representation is equivalent to the property of positive definiteness of both the observability and controllability Grammian matrices if these ones are well defined, i.e. if the representation has the property of asymptotic stability. From the mathematical correctness point of view there is no reason to require such a property in general case. Hence there is a need to separate the minimality conditions from the asymptotic stability conditions. It is well known that the minimality of linear time-invariant system representations can instead of Grammians be tested independently of stability property using rank conditions of the controllability and observability matrices H_c and H_o defined by the expressions

$$H_c = \left[B,\ AB,\ A^2B,\ ...,\ A^{n-1}B\right] \qquad (45)$$

$$H_o = \left[C^T,\ A^TC^T, (A^T)^2C^T,\ ...,\ (A^T)^{n-1}C^T\right] \qquad (46)$$

It is easy to show that for any two state equivalent state representations the following relations hold

$$\bar{H}_c = T\ H_c \qquad (47)$$

$$\bar{H}_o = (T^{-1})^T\ H_o \qquad (48)$$

and can be used for *determination of the state transformation* Equation (44) if the internal structure of both the equivalent representations is known.

7. STATE ENERGY ADITIVITY

Consider the time-invariant system representations Eqns.(37), (38); the finite controllability Grammian matrix $W_c(t)$ at time t is defined as follows

$$W_C(t) = \int_0^t e^{A\tau}.B.B^T.e^{A^T\tau}\, d\tau \qquad (49)$$

and has two important properties [4].

First, $W_c^T(t) = W_c \ge 0$ and secondly, the columns of $W_c(t)$ span the controllable space, i.e. the image of $W_c(t)$ equals to the image of the controllability matrix H_c :

$$\operatorname{im}\left[W_c(t)\right] = \operatorname{im}\left[H_c(A, B)\right] \qquad (50)$$

It can be shown that the state defined by the chosen structure of matrices A, B is controllable if, and only if $W_c(t)$ is positive

definite for some $t > 0$. If the controllability Grammian matrix is invertible, then the minimum energy input signal $u(.)$ exists and the *minimum input signal energy* corresponding to the state transfer from the zero state to $x(t)$ is known to be given by

$$\text{Energy}\{u(.)\} = x^T(t)\, W_c^{-1}\, x(t), \qquad t > 0 \tag{51}$$

Similarly, using the observability Grammian matrix $W_o(t)$, the output signal energy at time t caused by the initial state $x(t_0)$ is known to be given by the quadratic form:

$$\text{Energy}\{y(.)\} = x^T(t_0) W_o(t)\, x(t_0), \qquad t_0 < t \tag{52}$$

where the finite observability Grammian $W_o(t)$ at time $t < +\infty$ is defined by the expression [6]

$$W_o(t) = \int_{t_0=0}^{t} e^{A^T\tau}.C^T.C.e^{A\tau}\,\mathrm{d}\tau \tag{53}$$

and has two important properties. First, $W_o^T(t) = W_o(t) \geq 0$.

Secondly, for positive values of t, the kernel of finite observability Grammian matrix is equal to the kernel of observability matrix H_o

$$\ker[W_o(t)] = \ker[H_o(A,C)] \tag{54}$$

In the case of standard infinite time observability Grammian, i.e. for $t \to \infty$ *the largest observation energy* produced by any given value of initial state $x(t_0)$ is given by

$$E_o[x(t_0)] = x^T(t_0) W_o\, x(t_0) \tag{55}$$

assuming the couple *(A, C)* is observable and *A* is such that asymptotic stability conditions are satisfied. In such case the Lyapunov's equation

$$A^T W_o + A W_o = -C^T C \tag{56}$$

which in fact expresses a form of *the state-output energy transfer balance relation, can be used for determination of the unknown output energy function* $E_o(t)$.

In the dual case of standard infinite time controllability Grammian, i.e. for $t \to \infty$ the *minimal input energy* required to transfer the zero initial state to any state $x(t_1)$ is given by

$$E_I[x(t_0)] = x^T(t_1).W_c^{-1}.x(t_1) \tag{57}$$

assuming the couple (A,B) is controllable and A is such that asymptotic stability conditions are satisfied. In such a case the Lyapunov's equation

$$A W_c + A^T W_c = -B B^T \tag{58}$$

which in fact expresses a form of *the input-state energy transfer balance relation, can be used as a tool for determination of the unknown input energy function* $E_I(t)$.

Remark 9: In general, both the energy functions above are generated by symmetric positive semi-definite matrices, which depend heavily on the chosen system representation structure.

From physically motivated *energy additivity principle* (EAP):

$$E[x(.)] = \sum_{i=1}^{n} E_i[x_i(.)], \qquad x_i(.) \in R \tag{59}$$

it follows that only such system representation structures can be *accepted as physically correct*, which are not in contradiction with EAP, or equivalently for which the *Grammian matrices, generated by state equivalent triplets {A, B, C} are diagonal and non-singular, i.e., for which:*

$$W = W_o \text{ or } W_c \Leftrightarrow W = \text{diag}\{\delta_1, \delta_2, ..., \delta_n\},\ \delta_i \neq 0 \tag{60}$$

Definition 6: *(Energy additivity property)*

Any *element of the group* $\wp$ of state equivalence transformations will be called *physically correct,* or physical structure preserving*, if the energy additivity property* (46) *is an invariant of the group* $\wp$.

As a consequence of mathematical and physical correctness requirements any reconstructed structure obeys the *causality principle, energy conservation principle* and has the *state minimality property*, but concerning the total *number of independent structure parameters* no explicit requirement has been postulated yet.

8. PARAMETER MINIMALITY

It is of crucial importance to *eliminate* not only redundant state variables, but *all redundant structure parameters*, too.

Definition 7: *(Parameter minimality)*

A correct internal structure has the property of *parameter minimality* if removing any of its structural parameters *decreases the generality* of equivalent external system representation.

Example 3: Let a class of external representations is given

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + ... + a_{n-1}\dot{y}(t) + a_n y(t) = 0 \tag{61}$$

For *full generality* a parameterization by n independent constants $a_1, a_2, ..., a_n$ is needed. For $n = 3$, it is possible to find a *physically correct internal structure of the matrix A*

$$\mathrm{A} = \begin{bmatrix} -\kappa_1^2, & \alpha_1, & \beta_1 \\ -\alpha_1 & 0 & \alpha_2 \\ -\beta_1 & -\alpha_2 & 0 \end{bmatrix} \tag{62}$$

induced by the (arbitrary) choice of the measured output

$$y(t) = \kappa_1 x_1(t) \quad \Leftrightarrow \quad \mathrm{C} = [\kappa_1, 0, ..., 0] \tag{63}$$

but it can be *overparameterized Fig. 5.*

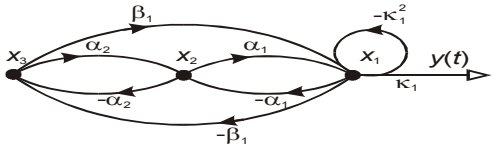


Fig.5. Correct but parameter non-minimal structure

Thus the question of *parameterically minimal structures* arises. It is easy to deduce that for the correct structure above the total number of N parameters is required for its parameterization:

$$N = \frac{1}{2}n(n-1)+1 \geq n, \qquad n = 1, 2, 3, ... \tag{64}$$

Intuitively, for $n = 3$ the right answer seems to be evident from the signal-flow graph *Fig. 5.,* i.e. that any one element of the set $\{\alpha_1, \alpha_2, \beta_1\}$ is redundant and can be removed (put equal to zero). To demonstrate that the *hypothesis is false,* the observability of *(A, C)* has to be tested*;* using standard technique we get the necessary and sufficient condition :

$$\det H_o \neq 0 \ \Leftrightarrow\ \kappa_1\alpha_2(\alpha_1^2 + \beta_1^2) \neq 0 \tag{65}$$

and *the right answer* is $\kappa_1 \neq 0$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_1 = 0$, or $\alpha_1 = 0$, and $\kappa_1 \neq 0$, $\beta_1 \neq 0$, $\alpha_2 \neq 0$, as shown at the *Fig. 6.*

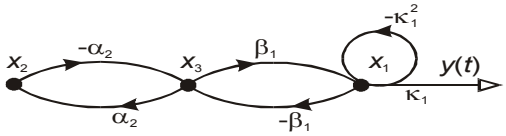


Fig 6:Parameter minimal equivalent correct structure

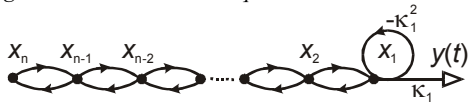


Fig 7: Parameter minimal correct chain structure

Remark 10: In spite of the fact that the *algebraic structures differ,* they represent *the same chain topological structure of the parameter minimal signal-flow graph Fig 7.*
The physical interpretation of the *internal interactions* is visualized for $\beta_i = 0$ by the *decomposition* at the *Fig. 8.* On the chain structure discussed above are based different classes of physically realizable filters, called asymptotic filters, as used for filter banks design in [4].

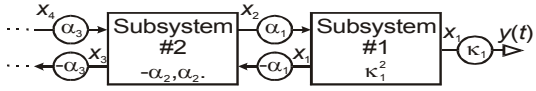


Fig.8: Physical interpretation of the system interactions

Now, if we put $\alpha_2 = 0$ then the natural requirement of sufficient generality will no more hold, because the equivalence transformation (44) does not exist for $\det T = 0$, and the resulting characteristic polynomial reduces to:

$$P(s) = s(s^2 + a_1 s + a_2), \quad a_1 = \kappa_1^2, \quad a_2 = \alpha_1^2 + \beta_1^2 \quad (66)$$

The structure analysis above can be summarized as:
Theorem 3: *(Parameter minimality)*
For a class of linear time-invariant finite dimensional *parametrically minimal* physically correct system representations the *state minimality, and sufficient generality are equivalent.*
It is obvious that *introducing non-vanishing inputs* the same analysis could be made using *dualized concepts.*

9. STATE VELOCITY SPACE AND GENERALIZED HESSENBERG STRUCTURE

It is challenging to find such a *structure of interactions* between the elements of the *constituent set* that the *intrinsic relationships* between fundamental system properties such as dissipativity and minimality will be clearly displayed.
In order to achieve the aim, it seems to be reasonable to characterize the *minimal dimension of the state velocity space structurally.* We start with a well known concept of the Hessenberg matrix :
Definition 8: *(Hessenberg structure of a matrix)*
Let A is a n-th order rectangular matrix. We say that the matrix A has the *Hessenberg structure* if it holds

$$1^o \qquad a_{i,j} = 0, \qquad j > i+1 \qquad (67)$$

$$2^o \quad a_{i,i+1} \neq 0, \text{ and } \quad sign(a_{i,i+1}) = 1 \qquad (68)$$

Definition 9: *(Hessenberg structure of a vector field)*
A vector field f has the *Hessenberg structure* if it holds

$$1^o \qquad \frac{\partial f_i}{\partial x_j} = 0, \qquad j > i+1 \qquad (69)$$

$$2^o \qquad \frac{\partial f_i}{\partial x_{i+1}} \neq 0, \qquad sign\left(\frac{\partial f_i}{\partial x_{i+1}}\right) = 1 \qquad (70)$$

Let n-th order nonlinear system representation is given

$$\Re\{S\}: \quad \dot{x}(t) = f[x(t)] + B\,u(t), \quad x(t_0) = x^0, \qquad y(t) = C\,x(t), \qquad (71)$$

and the matrices B and C have the form

$$C = [c_1, 0, \ldots, 0\ ], \qquad (72)$$

$$B^T = [\ 0, 0, \ldots, b_n\ ] \qquad (73)$$

Definition 10: *(Generalized Hessenberg structure of a system)*
We say that a *system representation* (15), (16) has the *Generalized Hessenberg structure if the* vector field f has the *Hessenberg structure*

$$1^o \qquad \frac{\partial f_i}{\partial x_j} = 0, \qquad j > i+1 \qquad (74)$$

$$2^o \qquad \frac{\partial f_i}{\partial x_{i+1}} \neq 0, \qquad sign\left(\frac{\partial f_i}{\partial x_{i+1}}\right) = 1 \qquad (75)$$

and in addition if it holds

$$3^o \quad c_1 \triangleq \frac{\partial h_1}{\partial x_1} \neq 0, \qquad sign\left(\frac{\partial h_1}{\partial x_1}\right) = 1 \qquad (76)$$

$$4^o \quad b_n \triangleq \frac{\partial f_n}{\partial u_n} \neq 0, \qquad sign\left(\frac{\partial f_n}{\partial u_n}\right) = 1 \qquad (77)$$

Remark 11: Notice that the Jacobian matrices have a *properly defined structure* motivated by the *system structure* corresponding to *the cascade connection of the elementary subsystems* according to the *Fig 9.*

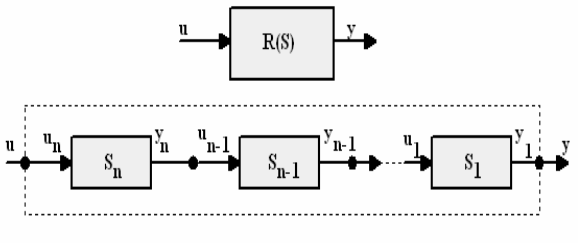


Fig..9. Generalized Hessenberg structure

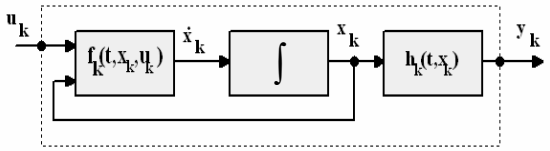


Fig.10. Internal structure of an elementary subsystem S_k

For the *internal structure of subsystems* S_k see *Fig.10.* The resulting system representation in *Generalized Hessenberg structure* is obviously *always controllable and observable,* i.e. *minimal* and is explicitly described by

$$\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, x_2), \\
\dot{x}_2 &= f_2(t, x_2, x_3), \\
\dot{x}_3 &= f_3(t, x_3, x_4), \\
&\cdots\cdots\cdots\cdots \\
\dot{x}_{n-1} &= f_{n-1}(t, x_{n-1}, x_n), \\
\dot{x}_n &= f_n(t, x_n) + u \\
y(t) &= h[t; x(t)] = x_1(t)
\end{aligned} \qquad (78), (79)$$

where the *set of external interactions* is given by

$$u(t) = u_n(t), \qquad (80)$$

$$y(t) = x_1(t) \qquad (81)$$

and the *set of internal interactions* is expressed by

$$u_i = x_{i+1}, \ i=1,2,\ldots,n-1 \qquad (82)$$

$$y_i = x_i, \quad i=1,2,\ldots,n \qquad (83)$$

10. BI-ORTHONORMAL BASIS OF STATE VELOCITY SPACE AND PHYSICALLY CORRECT STRUCTURE

In order to specify the *physically correct internal system structure* in the sense of *energy conservation principle validity* [3] we introduce a *structural representation*

$$\Re^{*}\{S\}: \quad Q\dot{x}(t) = A^{*}x(t) + B^{*}u(t), \qquad y(t) = C^{*}x(t) \tag{84}$$

Let us assume that *each elementary subsystem* S_k of the constituent set *is dissipative, i.e. it holds*

$$\forall i: \quad \frac{\partial f_i}{\partial x_i} < 0, \qquad i = 1, 2, ...n \tag{85}$$

Then the *simplest form of the structural matrix* A^{*} *in the Generalized Hessenberg representation* reads

$$A^{*} = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ : & : & : & ::: & : & : & : \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix} \tag{86}$$

Now, let the *structural matrices* Q, B^{*}, C^{*} be given by

$$Q = \begin{bmatrix} 1 & 0 & & 0 & 0 \\ -1 & 1 & & 0 & 0 \\ 1 & -1 & & . & . \\ -1 & 1 & \dots & . & . \\ . & -1 & . & 0 & 0 \\ . & . & . & 1 & 0 \\ . & . & . & -1 & 1 \end{bmatrix}, B^{*} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix}, (C^{*})^{T} = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix} \tag{87}$$

where the columns $\boldsymbol{q_1, q_2, \dots, q_n}$ of the matrix Q form a *biorthonormal basis* in the *state velocity space* given by

$$\boldsymbol{q_k + q_{k+1} = e_k}, \; k = 1,2, \dots, n\text{-}1, \; \boldsymbol{q_n = e_n} \tag{88}$$

Because Q is *always invertible*, we have

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & \dots & 1 & 0 \\ 0 & 0 & . & \dots & 1 & 1 \end{bmatrix} \tag{89}$$

and a resulting *generic structure of the matrix A* follows

$$A = Q^{-1}A^{*} = \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ : & : & : & : & ::: & : & : \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix} \tag{90}$$

11. STRUCTURE PARAMETERIZATION

Our goal is to specify a class of strictly causal system representations for which a form of energy conservation such as the Generalized Tellegen's principle holds. We start with the hypothesis that *it is not the physical energy by itself*, but only a *measure of distance from the system equilibrium to the actual state x(t)*, what is needed for this aim. Thus, instead of the physical energy a *metric* $\rho[x(t), x^{*}]$ will be defined in a proper way, and for an *abstract energy E(x)* we then put formally:

$$\mathrm{E(x)} \triangleq \frac{1}{2}\rho^{2}\left[\mathrm{x(t)},\ \mathrm{x}^{*}\right] = \frac{1}{2}\| x(t) - x^{*} \|^{2} \tag{91}$$

It has been shown in [3], [4], that the resulting *state equivalent system representation in dissipation normal form, corresponding to the derived generic structure* (88), (90) is described by a triple of matrices $\{ A, B, C \}$ as follows

$$A = \begin{pmatrix} -\alpha_1, & \alpha_2, & 0, & 0, & \cdots, & 0, & 0 \\ -\alpha_2, & 0, & \alpha_3, & 0, & \cdots, & 0, & 0 \\ 0, & -\alpha_3, & 0, & \alpha_4 & \cdots, & 0, & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots, & \vdots & \vdots \\ 0, & 0, & 0, & 0, & \cdots -\alpha_{n-1}, & 0, & \alpha_n \\ 0, & 0, & 0, & 0, & \cdots 0, & -\alpha_n & 0 \end{pmatrix} \tag{92}$$

$$C^{T} = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \tag{93}$$

It is easy to show that the set of *real design parameters* α_i, γ, β_i must satisfy the following fundamental *consistency conditions:*

1. $\forall i, i \in \{2, 3, ..., n\} : 0 \neq \alpha_i, 0 < \alpha_1 \Leftrightarrow$
   for structural asymptotic stability
2. $\forall i, i \in \{2, 3, ..., n\} : 0 \neq \alpha_i, \gamma \neq 0, \exists i : \beta_i \neq 0 \Leftrightarrow$
   for structural minimality

The *generic internal structure* of an *n*-th order continuous-time strictly causal system in *dissipation normal form* is shown at the *Fig. 11.*

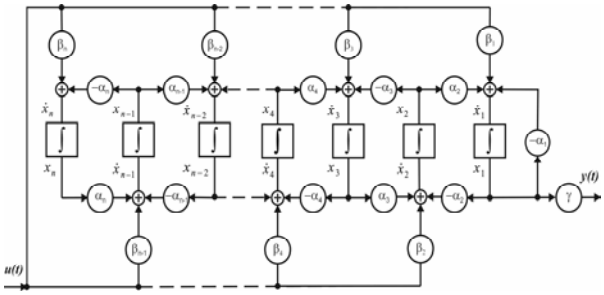


Fig. 11. Physically correct state space representation of continuous-time strictly causal system

12. DISSIPATIVITY AND STABILITY ANALYSIS

Example 4. (Stability analysis of a linear system)
Let the *n*-th order system representation is given by the linear differential equation with constant coefficients

$$y^{(6)} + a_1 y^{(5)} + \dots + a_4 \ddot{y}(t) + a_5 \dot{y}(t) + a_6 y(t) = 0 \tag{94}$$

with characteristic polynomial

$$P(s) = s^6 + a_1 s^5 + a_2 s^4 + \ldots + a_4 s^2 + a_5 s + a_6 \quad (95)$$

The *matrix A in the dissipation normal form* is given by

$$A = \begin{bmatrix} -\alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 \\ -\alpha_2 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & -\alpha_3 & 0 & \alpha_4 & 0 & 0 \\ 0 & 0 & -\alpha_4 & 0 & \alpha_5 & 0 \\ 0 & 0 & 0 & -\alpha_5 & 0 & \alpha_6 \\ 0 & 0 & 0 & 0 & -\alpha_6 & 0 \end{bmatrix} \quad (96)$$

and the corresponding *representation* reads

$$\begin{aligned} \Re(S): \quad \dot{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_2 x_2(t) \\ \dot{x}_2(t) &= -\alpha_2 x_1(t) + \alpha_3 x_3(t) \\ \dot{x}_3(t) &= -\alpha_3 x_2(t) + \alpha_4 x_4(t) \\ \dot{x}_4(t) &= -\alpha_4 x_3(t) + \alpha_5 x_5(t) \\ \dot{x}_5(t) &= -\alpha_5 x_4(t) + \alpha_6 x_6(t) \\ \dot{x}_6(t) &= -\alpha_6 x_5(t) \\ y(t) &= \gamma x_1(t) \end{aligned} \quad (97)$$

It follows that the parameters a_i, $i \in \{1, 2, ..., 6\}$ are given by

$$\begin{aligned} a_1 &= \alpha_1 \\ a_2 &= \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 \\ a_3 &= \alpha_1(\alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2) \\ a_4 &= \alpha_2^2(\alpha_4^2 + \alpha_5^2 + \alpha_6^2) + \alpha_3^2(\alpha_5^2 + \alpha_6^2) + \alpha_4^2\alpha_6^2 \\ a_5 &= \alpha_1\alpha_3^2(\alpha_5^2 + \alpha_6^2) + \alpha_1\alpha_4^2\alpha_6^2 \\ a_6 &= \alpha_2^2\alpha_4^2\alpha_6^2 \end{aligned} \quad (98)$$

Recall that the *necessary and sufficient condition for existence of the unique equilibrium state* $x^* = 0$ is given by

$$\det A = a_6 = \alpha_2^2\alpha_4^2\alpha_6^2 \neq 0 \quad (99)$$

From the existence of a unique equilibrium state point of view, the *dissipation parameter* α_1, as well as *interaction parameters* α_3, α_5 can be chosen *arbitrary*. For energy function *E* it holds:

$$E[x(t)] = \tfrac{1}{2}\rho^2[x(t),0] = \tfrac{1}{2}\|x(t)\|^2 = \tfrac{1}{2}\sum_{i=1}^{n} x_i^2(t) \quad (100)$$

$$1^o \quad E(x) = 0 \Leftrightarrow x(t) = x^*, \ (x^* = 0)$$

$$2^o \quad x_i(t) \in R \Leftrightarrow x_i^2(t) \geq 0 \Rightarrow E(x) > 0 \Leftrightarrow x(t) \neq x^*$$

For the *derivative of the state energy function E(x) along the system representation* (97) we get

$$\left.\frac{dE(t)}{dt}\right|_{\Re\{s\}} = -\alpha_1 x_1^2(t) = -\frac{\alpha_1}{\gamma^2} \cdot y^2(t) \quad (101)$$

where γ is a *real output scaling parameter*

$$0 < \gamma < \infty \quad (102)$$

Thus, for *non-zero output dissipation power* $y^2(t)$ the *signal energy conservation principle* holds if and only if:

$$P(t) = y^2(t) \quad \Leftrightarrow \quad \alpha_1 = \gamma^2 > 0 \quad (103)$$

Remark 12: Notice that the *dissipation parameter* α_1 is *the only element* of the matrix *A,* which *sign separates* the *system dissipativity* from its *anti-dissipativity.*

The *critical value* of $\alpha_1 = 0$, corresponds to the *system conservativity* and *separates stability* of the *equilibrium state* from its *instability*.

Remark 13: Notice, that if we put $\alpha_5 = 0$, then the state variables x_i, *i = 5, 6* become *unobservable by the output y* ; thus *only the first isolated subsystem* with the state variables x_i, *i = 1,2,3,4,* which is *observable, will be asymptotic stable*, while the *second one will oscillate on the constant energy level*, *(see Fig. 13.c for energy evolution).* Similarly, if we put $\alpha_3 = 0$, then the state variables x_i, *i = 3,4,5,6* become *unobservable by the output y, and only the observable subsystem*

$$\begin{aligned} \dot{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_2 x_2(t) \\ \dot{x}_2(t) &= -\alpha_2 x_2(t) \\ y(t) &= \gamma x_1(t) \end{aligned} \quad (104)$$

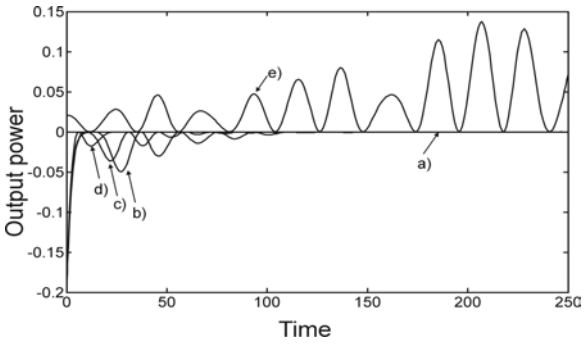


Fig.12. Evolution of the output power P(t)

a) $\alpha_1 = 0$, conservativity, b) $\alpha_1 > 0$, $\alpha_3 = 0$, stability, c) $\alpha_1 > 0$, $\alpha_5 = 0$, stability, d) $\alpha_1 > 0$, $\alpha_k \neq 0$, k = 2,3,…,n, asymptotic stability, e) $\alpha_1 < 0$, α_k, k = 2,3,…,n, arbitrary, instability

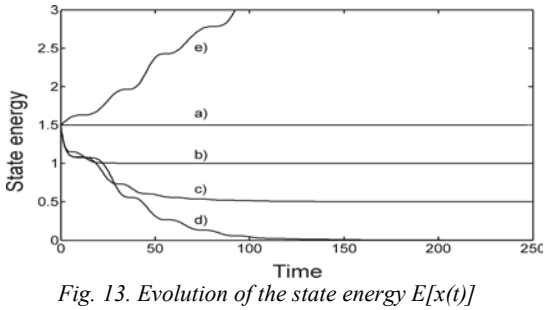


Fig. 13. Evolution of the state energy E[x(t)]

If needed, we can determine the parameters α_i, *i* = 1, 2,…, n from the Eqn. (98) then we get:

$$\begin{aligned} \alpha_1 &= a_1 = \Delta_1, \\ \alpha_2 &= \sqrt{\frac{a_1 a_3 - a_3}{a_1}} = \sqrt{\frac{\Delta_2}{\Delta_1}} \\ \alpha_3 &= \sqrt{\frac{a_1 a_2 a_3 - a_3^2 - a_1^2 a_4}{(a_1 a_2 - a_3) a_1}} = \sqrt{\frac{\Delta_3}{\Delta_2 \Delta_1}} \\ \alpha_k &= \sqrt{\frac{\Delta_k \Delta_{k-3}}{\Delta_{k-2} \Delta_{k-1}}}, \quad k = 4, 5, 6, \ldots, n \end{aligned} \quad (105)$$

where the *new parameters* Δ_k , *k= 1, 2, ..., n* can easily be recognized as *diagonal minors* of the well known *Hurwitz determinant* Using the Eqns. (40), (29) together with the requirement $\alpha_k \in R$, the following set of *equivalent necessary and sufficient conditions of the asymptotic stability* can be obtained in general finite dimensional case:

$$\begin{aligned}
&\alpha_1 \in R,\ \alpha_1 > 0 \quad \Leftrightarrow \quad \Delta_1 > 0 \\
&\alpha_2 \in R,\ \alpha_2 \neq 0 \quad \Leftrightarrow \quad \frac{\Delta_2}{\Delta_1} > 0 \\
&\alpha_3 \in R,\ \alpha_3 \neq 0 \quad \Leftrightarrow \quad \frac{\Delta_3}{\Delta_1 \Delta_2} > 0 \\
&\alpha_n \in R,\ \alpha_n \neq 0 \quad \Leftrightarrow \quad \frac{\Delta_{n-3}\Delta_n}{\Delta_{n-2}\Delta_{n-1}} > 0
\end{aligned} \tag{106}$$

The resulting conditions are obviously *equivalent to the set of the well-known Hurwitz stability conditions:*

$$\Delta_k > 0,\ k = 1, 2, \ \ldots, n \tag{107}$$

It means that existing linear *algebraic methods* for stability analysis can be seen as a *special case* of methods based on the proposed signal energy-metric approach. Moreover the proposed state energy interpretation makes it possible to gain a better insight into the classical results of stability theory.

13. GENERATION OF LYAPUNOV FUNCTIONS

Example 5. Let's consider a linear system given in the form

$$y^{(4)}(t) + a_1 y^{(3)}(t) + a_2 \ddot{y}(t) + a_3 \dot{y}(t) + a_4 y(t) = 0 \tag{108}$$

gained *by an approximate linearization* procedure, and let the state variables be defined by the standard way as follows

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}, \quad x_4 = y^{(3)} \tag{109}$$

Then the *observability matrix* is given by $H_o = I$, while the *observability matrix* $\overline{H}_o$ of the state equivalent representation $\Re\{S\}$ is *triangular and invertible*. It is easy to show that the Lyapunov function *V* can be computed by *isometric transformations* of the state space coordinates:

$$V\left[x(t)\right] = \tfrac{1}{2} x^T(t)\left[\overline{H}_o^T . \overline{H}_o\right]^{-1} . x(t) \tag{110}$$

and it can be explicitly expressed

$$\begin{aligned}
V = \frac{1}{2}\Bigg[& x_1^2 + \left(\frac{\alpha_1}{\alpha_2} x_1 + \frac{1}{\alpha_2} x_2\right)^2 + \\
& + \left(\frac{\alpha_2}{\alpha_3} x_1 + \frac{\alpha_1}{\alpha_2 \alpha_3} x_2 + \frac{1}{\alpha_2 \alpha_3} x_3\right)^2 + \ . \ . \ . \Bigg]
\end{aligned} \tag{111}$$

14. NON-LINEAR STABILITY ANALYSIS

Example 6. (Generalized Van der Pol system)

Let us consider a simple *non-linear system given* by

$$\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right]\dot{y}(t) + a_2 y(t) = 0 \tag{112}$$

If *C* is defined by $C = [\gamma, 0]$, and *A(x)* is defined by the *non-linear dissipation normal form*

$$A(x_1, x_2) = \begin{bmatrix} -\varepsilon\left[\alpha - \frac{1}{3}\beta x_1^2\right], & \sqrt{a_2} \\ -\sqrt{a_2}, & 0 \end{bmatrix} \tag{113}$$

then the system representation is *locally observable iff*

$$\gamma \neq 0, \ \ a_2 > 0 \tag{114}$$

and the *signal energy conservation principle* gives

$$\left.\frac{\mathrm{d}E(t)}{\mathrm{d}t}\right|_{\Re(s)} = -P \leq 0, \ \ P = \varepsilon\left[\alpha - \tfrac{1}{3}\beta x_1^2\right] x_1^2 \tag{115}$$

It follows that the *unique zero equilibrium state* $x^* = 0$ is *asymptotically stable in the region* $D \subset X \subset R^2$

$$D = \left\{ x_1, x_2 : |x_1| < \sqrt{\frac{3\alpha}{\beta}} \text{ and } x_1^2 + x_2^2 < \frac{3\alpha}{\beta} \right\} \tag{116}$$

if $\varepsilon > 0, \ \alpha > 0, \ \beta > 0, a_2 > 0$.

Example 7. (Generation of non-quadratic Lyapunov functions) Let the same *non-linear system* be *given*

$$\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right]\dot{y}(t) + a_2 y(t) = 0 \tag{117}$$

but instead of the *structure the* state vector $x(t)$ is defined by

$$x_1 = y, x_2 = dy / dt \tag{118}$$

Then the corresponding system representation is *structurally observable* with the *observability matrix* $H_o = I$, and from the *signal energy conservation principle*

$$\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{\Re(s)} = -P \leq 0, \ \ P = \varepsilon\left[\alpha - \tfrac{1}{3}\beta x_1^2\right] x_1^2 \tag{119}$$

a *unique Lyapunov function* $V(x)$ can be determined by *isometric transformations*. For $\alpha = \beta = a_2 = 1$ we get

$$\begin{aligned}
V(x) = \tfrac{1}{2}\Big[& \tfrac{1}{9}\varepsilon^2 x_1^6 - \tfrac{2}{3}\varepsilon^2 x_1^4 + (1+\varepsilon^2) x_1^2 - \\
& - \tfrac{2}{3}\varepsilon x_1^3 x_2 + 2\varepsilon x_1 x_2 + x_2^2 \Big]
\end{aligned} \tag{120}$$

and for *linear conservative case* ($\varepsilon = 0$) it reduces to

$$V(x) = \tfrac{1}{2}\left(x_1^2 + x_2^2\right) \tag{121}$$

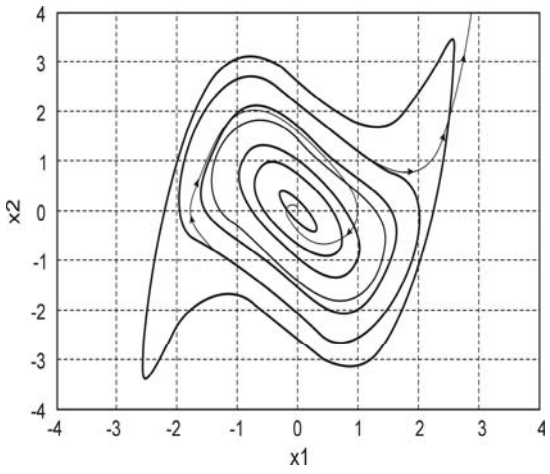


Fig. 14. Phase portrait of typical system trajectories, estimated and actual domain of attraction, and constant levels of the state energy E[x(t)]=V[x(t)], (for $\varepsilon = 1, \ \alpha = \beta = a_2 = 1$)

Example 8. (Lewis servo-system)
Let us consider another well known *non-linear system* given by

$$\ddot{y}(t) + a_1\left[1-\beta y(t)\text{sign}[y(t)]\right]\dot{y}(t)+a_2 y(t) = 0 \quad (122)$$

$$A(x_1,x_2) = \begin{bmatrix} -a_1\left[1-\beta x_1(t)\text{sign}[x_1(t)]\right], & \sqrt{a_2} \\ -\sqrt{a_2}, & 0 \end{bmatrix} \quad (123)$$

with $\alpha_2 > 0$, and α_1, β as arbitrary real design parameters.
A *unique Lyapunov function* $V(x)$ can be determined by *isometric transformations as the state energy E(x.).* We get:

$$E\left[x_1,x_2\right] = \frac{1}{4}a_1^2\beta^2 x_1^4 - a_1^2\beta x_1^3\text{sign}\left[x_1(t)\right] - \\ - a_1\beta x_1^2 x_2\text{sign}\left[x_1(t)\right] + (a_1^2+1)x_1^2 + 2a_1x_1x_2 + x_2^2 \quad (124)$$

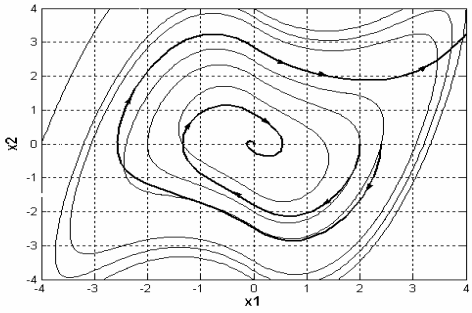


Fig. 15. Phase portrait of typical system trajectories, estimated domain of attraction, and constant levels of the state energy E[x(t)]=V[x(t)], (for α_1 =1, $\beta=\alpha_2$ = 1)

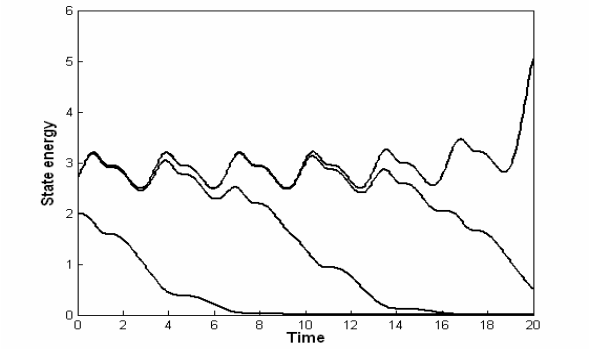


Fig. 16. Evolution of the state energy E[x(t)], for $\alpha_1>0$, i.e. for asymptotically stable zero equilibrium state, and for periodic solution - unstable limit cycle.

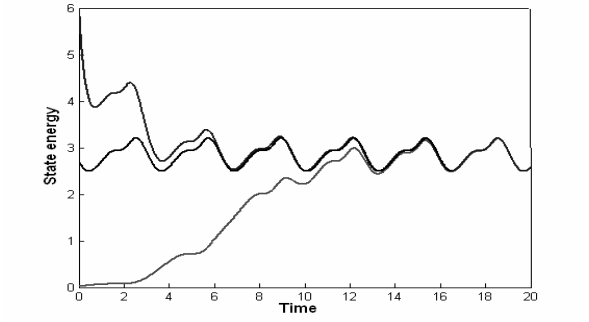


Fig. 17. Evolution of the state energy E[x(t)], for $\alpha_1 < 0$, i.e. for unstable zero equilibrium state, and for periodic solution - asymptotically stable limit cycle.

The dissipation power *P(t)* is given by

$$\forall a_2 \geq 0:\ \frac{dE}{dt} = -2a_1.x_1^2\left[1-\frac{1}{2}\beta x_1\text{sign}[x_1(t)]\right] \quad (125)$$

For *linear case* (β= 0) and for $\alpha_2 = 1$ the energy reduces to

$$E\left[x_1,x_2\right] = (a_1^2+1).x_1^2(t) + 2a_1x_1(t).x_2(t) + x_2^2(t) \quad (126)$$

Example 9. (Generation of chaos in a strictly causal system)
Let a 4th order system be given by

$$\begin{aligned} \dot{x}_1 &= -\alpha_1 x_1 + \alpha_2 x_2 \\ \dot{x}_2 &= -\alpha_2 x_1 + \alpha_3 x_3 \\ \dot{x}_3 &= -\alpha_3 x_2 + \alpha_4 x_4 \\ \dot{x}_4 &= -\alpha_4 x_3 \end{aligned} \qquad \alpha = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}-\left[1-10x_2(t)\text{sign}[x_2(t)]\right]\\ 1\\ 1\\ 1.65\end{bmatrix} \quad (127)$$

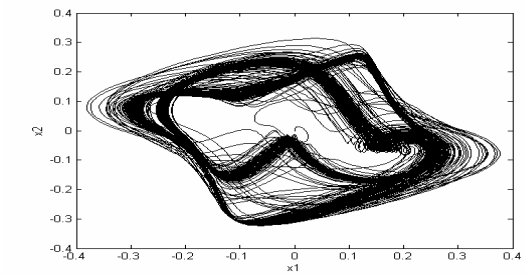


Fig. 18. 2-D projection chaotic system trajectories

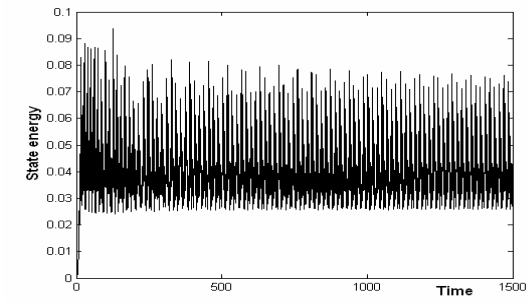


Fig. 19. Evolution of non-periodic- chaotic state energy E[x(t)]

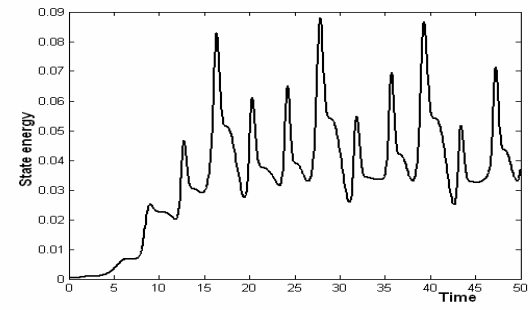


Fig. 20. Evolution of the chaotic state energy E[x(t)] - zoom

From diagrams in the *Fig.13., Fig.16., Fig.17.,* and the *Fig.20.* it is obvious that in all cases the local energy changes can be characterized either as monotone decreasing or as monotone increasig on some subintervals of the time interval [0, ∞). The only difference is that in the case of chaotic behavior the length of these dissipativity/anti-dissipativity intervals is totally irregular, while in case of non-linear but periodic behavior the length of dissipation and/or antidissipation intervals is constant, given by the period of oscillations and by the specific form of nonlinearity. The interpretation of dissipative, conservative and anti-dissipative linear cases, as well as aperiodic asymptotic transient solutions in non-linear systems, is quite plausible.

15. PHYSICAL STRUCTURE RECONSTRUCTION

Example 10. (Reconstruction of physical structure for n=4)
Let's consider a linear 4-th order system with 1 informational output and 3 independent input signals, described by an external representation. Find an internal structure of the given system in such a way that the resulting physical system representation will be physically as well as mathematically correct.
Let's assume that the system is known to be asymptotic stable with 2 dissipative and 4 energy accumulation elements. In such a case the physically correct internal system representation $\Re\{S\}$ can be described as follows:

$$\Re\{S\}: dx(t)/dt = Ax(t) + Bu(t), \quad (128)$$

$$y(t) = Cx(t), \quad (129)$$

$$u_3(t) = Kx(t), \quad (130)$$

and the matrices *A, B, C, K* are given in the form *(Fig. 21.)*

$$A = \begin{bmatrix} -\alpha_1 & \alpha_2 & 0 & 0 \\ -\alpha_2 & 0 & \alpha_3 & 0 \\ 0 & -\alpha_3 & 0 & \alpha_4 \\ 0 & 0 & -\alpha_4 & 0 \end{bmatrix}, B = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, u = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \quad (131)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & -k_4 \end{bmatrix} \quad (132)$$

and the abstract form of state energy is defined by

$$E(x) \triangleq \frac{1}{2}\rho^2\left[x(t),\, 0\right] = \frac{1}{2}\{x_1^2 + x_2^2 + x_3^2 + x_4^2\} \quad (133)$$

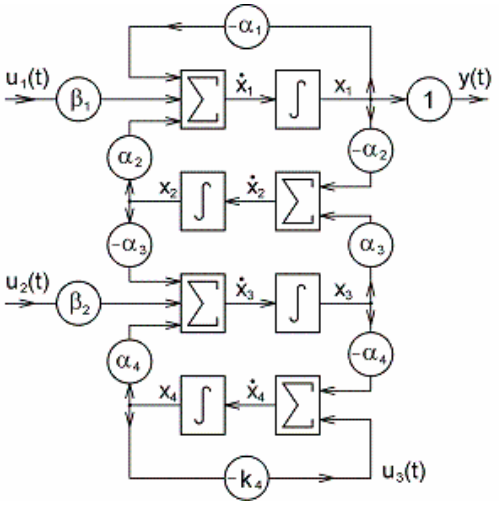


Fig. 21. Physically correct structure of the given representation

Let us consider an electrical circuit (for example) as a physical realization of the system under consideration. If the inputs $u_1(t)$, $u_2(t)$, are defined by voltages of an electrical circuit

$$u_1(t) = u_{01} - u_{02}, \quad u_2(t) = u_{03} - u_{02} \quad (134)$$

and the state variables x_1, x_2, x_3, x_4 are defined as currents and voltages of the circuit then, for a proper state scaling transformation

$$x_1 = \sqrt{L_1}\, i_1,\ x_2 = \sqrt{C_2}\, u_2,\ x_3 = \sqrt{L_3}\, i_3,\ x_4 = \sqrt{C_3}\, u_3 \quad (135)$$

the total system energy in new (physical) state variables takes the form:

$$\mathrm{E(x)} = \frac{1}{2}\{L_1 i_1^2 + C_2 u_2^2 + L_3 i_3^2 + C_3 u_3^2\} \quad (136)$$

and the standard form of energy conservation principle holds.
One of possible physical structures is shown in the *Fig. 22.*

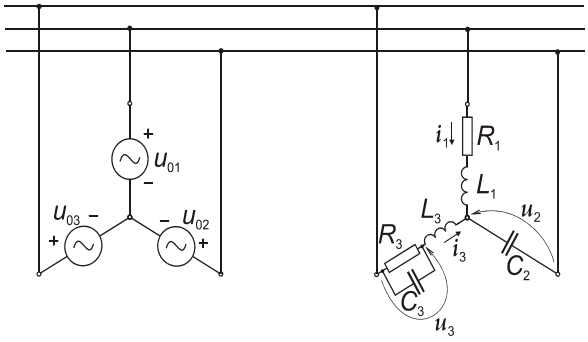


Fig. 22.Physical example of the reconstructed structure

16. CONCLUSIONS

In the present paper basic concepts concerning new problem of system structure reconstructability, dissipativity, conservativity, state minimality, parameter minimality, internal stability, instability and chaos have been examined from a unified structural point of view. Both the linear as well as non-linear state-output system representations are discussed.
The work relates some fundamental attributes of real-world situations such as causality, mathematical and physical correctness and different forms of conservation laws to specific notions and results of the electrical circuits theory, as well as to some basic approaches and concepts of general system theory, such as state minimality, signal power, signal energy, equivalence relations, controlability and observability

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17. REFERENCES

[1] P. Penfield, Jr., S. Spence, S. Dunker, "Tellegen´s Theorem and Electrical Networks" **Research Monograph No. 58, Cambridge, MIT Press,** 1970.

[2] D. Mayer, J. Hrusak, "On Correctness and Asym-ptotic Stability in Causal Systems Theory", **Proc. of 7th World Multiconf. SCI, Vol. XIII, pp. 355-360, 2003.**

[3] J. Hrusak, D. Mayer, M.Stork, "On System Structure Reconstruction Problem And Tellegen-Like Relations", **Proc. of 8th World Multiconf.,SCI, Vol. VIII, pp. 373-378, Florida, USA, 2004.**

[4] J. Hrusak, V. Cerny, D.Panek, "A New Approach to System Structure Reconstruction Based On A Generalized Tellegen Principle", **Preprints of 16th IFAC World Congress, July 3-8, Prague, CZ, 2005.**

[5] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems", **J. SIAM on Control** , Series A, Vol.1., No.2, p. 152, 1963.

[6] J.C. Willems, "Paradigms and Puzzles in the Theory of Dynamical Systems", **IEEE Trans. on Automatic Control**, Vol. 36, No. 3, pp. 259-294, 1991.

[7] H. W. Sorenson, "Parameter Estimation, Principles and Problems", **Marcel Dekker, Inc., N. Y., 1980.**