Multigraph Decomposition Into Multigraphs With Two Underlying Edges

Miri Priesler
School of Engineering, Ruppin Academic Center
Ruppin, Emek hefer 40250, ISRAEL

and

Michael Tarsi
School of Computer Science, Sackler Faculty of Exact Sciences, Tel-Aviv university
Tel-Aviv, 6997, ISRAEL

ABSTRACT

Due to some intractability considerations, reasonable formulation of necessary and sufficient conditions for decomposability of a general multigraph G into a fixed connected multigraph H, is probably not feasible if the underlying simple graph of H has three or more edges. We study the case where H consists of two underlying edges. We present necessary and sufficient conditions for H-decomposability of G, which hold when certain size parameters of G lies within some bounds which depends on the multiplicities of the two edges of H. We also show this result to be "tight" in the sense that even a slight deviation of these size parameters from the given bounds results intractability of the corresponding decision problem.

Keywords: Multigraphs , Decompositions, Stars, Multistars, Intractability.

1. INTRODUCTION

Given two graphs H and G, an H-decomposition of G is a partition of the edge set of G into disjoint isomorphic copies of H. The study of Graph decomposition started back at the mid19th century, with the seminal concept of *Steiner triple systems* [8], and has since become the subject of some hundreds of research papers, with active research still carried out today. R. Wilson's fundamental theorem [9] states that for any fixed graph H there exists an H-decomposition of the complete graph K_n if the obvious necessary divisibility conditions hold and n is *large enough*. A considerable amount of research was indeed devoted to thoroughly studying the existence of H decompositions of complete graphs for specific graphs H, such as: some small graphs, complete graphs, complete multipartite graphs, paths and cycles (a finite problem for every fixed graph H, in light of Wilson's theorem). For a review of methods and results see e.g. [3] and [2].

Hopes for similar accurate results where H decomposition of a general graph G is considered are slim, due to the following negative result:

Theorem 1.1 *Deciding whether there exists an H-decomposition of an input graph G is* NP-*complete for any fixed simple graph H which contains a connected component with at least 3 edges*

The above was conjectured by I. Holyer [5] on 1981 and proved sixteen years later in [4]. On the other hand, the existence of a polynomial time algorithm to decide H-decomposability of an input G, where every component of H consists of at most two edges was proved (though not in terms of an explicit necessary and sufficient condition) in [1].

In this research we study *Multigraph decomposition*, that is the case where multiple edges are allowed in both graphs H and G. Although Theorem 1.1 was not (yet?) generalized to multigraphs, a graph decomposition decision problem most probably remains at least as hard when extended to multigraphs. Furthermore, we have managed to prove the following intractability results [7]:

Theorem 1.2 *Deciding the decomposability of an input multigraph G with a constant multiplicity λ on all its edges, into the star $K_{1,t}$ is NP-Complete for every fixed λ and $t \geq 3$.*

Theorem 1.3 *Deciding H-decomposability of an input multigraph G into any fixed multistar (a multigraph whose underlying simple graph is $K_{1,t}$, with any sequence of positive multiplicities on its t edges) with at least three underlying edges, is NP-Complete.*

In an attempt to find the conditions for decomposability of a general "input" multigraph G into a "fixed" connected multigraph H, serious hopes for results are limited, in light of the theorems above, to the case were H consists of two underlying edges.

Quite surprisingly we found out this limited setting to be rather involved, producing somewhat unexpected results: In a previous article [6], we considered the simplest case, where $H=S^{1,2}$ is a multigraph on an underlying $K_{1,2}$ with multiplicity 1 on one edge and 2 on the other, and G is a multigraph on any underlying simple graph with a constant multiplicity λ on all its edges. We gave necessary and sufficient condition for such a decomposition to exist if $\lambda \neq 2$ and $\lambda \neq 5$. We also showed that similar conditions for $\lambda = 2$ and for $\lambda = 5$ do most probably not exist, by proving the corresponding decision problems to be NP-complete.

In Section 2 of this article we investigate the decomposition of a general multigraph G into $S^{\alpha,\beta}$-an underlying $K_{1,2}$ with multiplicities α and β. We show some necessary divisibility conditions to be also sufficient if certain size parameters of G lie between certain bounds which depend on α,β and $\frac{\alpha}{\beta}$. We then show in Section 3 this result to be "best possible" in the sense that the corresponding decision problem becomes NP-complete when the relevant size requirements are not met.

The following terminology sets the frame for a more formal and rigorous treatment of the subject.

Notation

- *A multigraph* (V,E,w), also denoted by (G,w), consists of a simple *underlying graph* G=(V,E) and a *multiplicity function* w: E $\rightarrow$ N, where N is the set of natural numbers (unless explicitly stated otherwise, the multiplicity of an edge is strictly positive).
- The multigraph on an underlying graph G with a constant multiplicity λ is denoted by $\lambda \cdot G$.
- When referring to a simple graph G as a multigraph, we mean$1 \cdot G$.

- An isomorphism between multigraphs is an isomorphism between their underlying simple graphs, which preserves edge multiplicity.
- A *subgraph* H of a multigraph G is a multigraph H whose underlying graph is a subgraph of that of G and its multiplicity function is dominated by the multiplicity function of G, i.e. the multiplicity of an edge in H does not exceed its multiplicity in G.
- An H- *subgraph* of G is a subgraph of a multigraph G, isomorphic to a multigraph H.
- Let G and H be two multigraphs. An H-decomposition of G is a set D of H-subgraphs of G, such that the sum of w(e) over all graphs in D which include an edge e, equals the multiplicity of e in G, for all edges e in G.
- An H- λ *decomposition* of a simple graph G is an H-decomposition of the multigraph $\lambda\cdot G$. If it exists we say that G is H-λ decomposable, or that G admits an H-λ decomposition.
- The t-*star*, S_t , (also commonly denoted by $K_{1,t}$) is a simple graph, consisting of t edges which share one common vertex, referred to as the *center* of the star, and are otherwise disjoint.
- The *multistar* $S^{w_1,w_2,...,w_t}$ is the multigraph, whose underlying graph is a t-star, and the multiplicities of its t edges are $w_1,...,w_t$.
- Associated with a fixed multigraph H is the H-decomposition computational problem: *Does an input multigraph M admit an H-decomposition?*
- In particular, associated with a fixed multigraph H and a natural number λ is the H-λ decomposition computational problem: *Does an input simple graph G admit an H-λ decomposition?*

2. $S^{\alpha,\beta}$-DECOMPOSITIONS OF MULTIGRAPHS WITH A NARROW INTERVAL OF LARGE MULTIPLICITIES

In this section we prove Theorem 2.1 which states that Some necessary divisibility conditions for $S^{\alpha,\beta}$-decomposability of a multigraph G are also sufficient if the multiplicities of all edges of G are large enough and yet the ratio between the largest and smallest multiplicities is bounded. For a certain class of trees, which we refer to as *odd regular trees*, there is also a lower bound on the number of underlying edges of G.

More definitions and some preparatory work are required for the detailed formulation of Theorem 2.1 and hence we leave it to a later stage.

Along the rest of this chapter we assume that α and β are relatively prime, otherwise α, β and the multiplicity function should first be divided by $gcd(\alpha,\beta)$. We also assume $\alpha>\beta$.

Binding together the $S^{\alpha,\beta}$-subgraphs which share a common center and considering a single edged multigraph decomposable, if and only if its multiplicity is 0, the following is clearly an equivalent way to look at an $S^{\alpha,\beta}$-decomposition of a multigraph which is not a star.

Definition *$S^{\alpha,\beta}$-decomposition of a multigraph M=(V,E,w), other than a star, is a two-variables function f, which assigns an integer $f_x(e)$ to each pair (x,e), where $x \in V$ is a vertex and e is an edge, incident with x, such that:*

1. *f_x(e)+f_y(e)=w(e) for every edge e=(x,y)\in E, and:*
2. *for every vertex $x \in V$, incident with edges $e_1,...,e_n$, the n-star with multiplicities $f_x(e_1),...,f_x(e_n)$, is $S^{\alpha,\beta}$-decomposable.*

As it is often the case when a decomposition problem is at hand, there are some simple necessary divisibility conditions for $S^{\alpha,\beta}$-decomposition, which become also sufficient when certain size parameters of the Input multigraph are large enough. Our method of constructing $S^{\alpha,\beta}$-decompositions deals with these two aspects in two separate phases: First we consider only the 'large enough' inequalities to construct a non-integer `approximation' which is then slightly modified until it fits the divisibility conditions.

We now state the necessary divisibility condition for $S^{\alpha,\beta}$-decomposition by considering its projection on the ring of integers modulo α^2 - β^2 .

2.1 $S^{\alpha,\beta}$-decomposition in the ring of integers modulo α^2- β^2

The divisibility conditions for an $S^{\alpha,\beta}$-decomposition are captured by the following definition:

Definition *A $Z_{\alpha,\beta}$-decomposition of a multigraph M = (V,E,w), is a collection D of $S^{\alpha,\beta}$-subgraphs of M, such that the sum of multiplicities of an edge e over all members of D which include e is congruent to w(e) modulo α^2 - β^2.*

Based on the argument which led to that Definition , we obtain the following equivalent definition for graphs which are not stars:

Definition *A $Z_{\alpha,\beta}$-decomposition of a multigraph M=(V,E,w), other than a star, is a two-variables function f, which assigns an integer $f_x(e)$ to each pair (x,e), where x is a vertex and e is an edge, incident with x, such that:*

1. *$f_x(e)+f_y(e) \equiv w(e)$ (modα^2 - β^2 (for every edge e = (x,y) $\in$ E, and:*
2. *for every vertex $x \in V$, incident with edges $e_1,...,e_n$, the n-star with multiplicities $f_x(e_1),...,f_x(e_n)$, is $Z_{\alpha,\beta}$-decomposable.*

An $S^{\alpha,\beta}$-decomposition of a 2-star on edges e_1,e_2 consists of a number, say a, of copies of $S^{\alpha,\beta}$, with multiplicity α on e_1 and β on e_2, and additional b copies where the multiplicities are switched. The multiplicities on the edges of a decomposable 2-star are hence $a\alpha + b\beta$ and $b\alpha + a\beta$. Let $Z_{\alpha,\beta}$ stand for the ring of integers modulo α^2 - β^2. We say that a pair $(p,q) \in Z^2_{\alpha,\beta}$ is *valid* if there exist a and b in $Z_{\alpha,\beta}$ such that $p = a\alpha + b\beta$ and $q = b\alpha + a\beta$.

A multiplicity function on the edges of a graph G naturally translates to a weight function on the vertices of its line graph, an S_2 subgraph of G is merely an edge of the line-graph and an $S^{\alpha,\beta}$ subgraph becomes an edge with weight α on one endvertex and β on the other. In light of that observation we define a $Z_{\alpha\beta}$-*factorization* of a graph L=(U,F) with a weight function w : $\rightarrow Z_{\alpha\beta}$ to be a function f which assigns a valid pair $(f_e(u),f_e(v))$ to every edge e = (u,v) $\in$ F, such that the sum of $f_e(u)$ over all edges e incident with a vertex u, equals w(u). A $Z_{\alpha\beta}$-*decomposition* of a multigraph is clearly a $Z_{\alpha\beta}$-factorization of its line-graph, with the same multiplicity (weight) function, and the existence of a $Z_{\alpha\beta}$-*decomposition* is necessary for the existence of an $S^{\alpha,\beta}$-decomposition.

Lemma 2,1 *A necessary and sufficient condition for $(p,q) \in Z^2_{\alpha,\beta}$ to be a valid pair is $\alpha p = \beta q$.*

Proof: Since α and β are relatively prime, both are multiplicative generators of $Z_{\alpha\beta}$ and division by α and by β is well defined. We use s to denote the quotient $\frac{\alpha}{\beta} \in Z_{\alpha\beta}$. The relation α^2 - β^2 leads to $s^{-1} = \frac{\beta}{\alpha} = \frac{\alpha}{\beta}$ = s. We restate $\alpha p = \beta q$ as q = sp (equivalently p= sq).

Necessity: If (p,q) is valid then there exist a and b such that $p = a\alpha + b\beta$ and $Q = a\beta + b\alpha = as\alpha + bs\beta = sp$.

Sufficiency: (p,sp) = ($\frac{p}{\alpha}$ α, $\frac{p}{\beta}$ β) is valid. □

Lemma 2.2 *A connected multigraph M = (V,E,w) admits a $Z_{\alpha\beta}$-decomposition if and only if:*

1. *$\sum_{e\in E} w(e)$ is divisible by $\alpha + \beta$.*
2. *If G is either a simple path, or an even circuit then $\alpha\sum_{e \text{ is odd}} w(e) \equiv \beta\sum_{e \text{ is even}} w(e)$ (mod α^2 - β^2) ,*

where `odd' and `even' refer to the location of e along the path (or even circuit, with an arbitrary first vertex).

The tighter restriction imposed on paths and even circuits is originated in these graphs being the only ones of which the line-graph is bipartite. Lemma 2.2 is proved, by induction, through $Z_{\alpha\beta}$-factorization of general graphs, rather than just line-graphs. Bipartite and non-bipartite graphs are dealt with separately. Within the scope of the following two propositions, arithmetics is again that of the ring $Z_{\alpha\beta}$:

Proposition 2.1 *Let L=(U,F,w) be a connected bipartite weighted (over the vertices) graph, where U is partitioned into U_1 and U_2 and every edge has one endvertex in U_1 and one in U_2. A necessary and sufficient condition for L to admit a $Z_{\alpha\beta}$-factorization is* $\alpha\sum_{x\in U_1} w(x) \equiv \beta\sum_{y\in U_2} w(y)$.

Proof: Again we set s = $\frac{\alpha}{\beta} \in Z_{\alpha\beta}$. Necessity immediately follows from Lemma 2.1. Sufficiency: Let u be a vertex of L such that L \ {u} is still connected and let v ∈ U be adjacent to u. Apply the Lemma to the smaller (induction) graph L \ {u} with the weight of v set to w(v)-sw(u). Then assign the valid pair (w(u),sw(u)) to the edge (u,v) to reach the required weights of u and v. □

Proposition 2.2 *Let L=(U,F,w) be a connected weighted graph, which is not bipartite. A necessary and sufficient condition for L to admit a $Z_{\alpha\beta}$ -factorization is* $\sum_{u\in U} w(u)$ *belongs to the ideal generated by α + β.*

Proof: Let us first observe that $\alpha^{-1}(\alpha+\beta) = 1+s$ and $\alpha(1+s) = \alpha+\beta$. The ideal generated by α + β is then also generated by s + 1. *Necessity*: Each valid pair contributes p + sp = p(s + 1) to $\sum_{u\in U} w(u)$. *Sufficiency*: Let us select a spanning bipartite subgraph L' of L (e.g. a spanning tree), where U is partitioned into U_1 and U_2 and every edge of L' has one endvertex in U_1 and one in U_2. Let us also select an edge (u,v) with both endvertices in the same side, say, in U_2. Define a = $\sum_{x\in U_1} w(x)$. If the condition set by Proposition 2.2 is met then there exists t ∈ $Z_{\alpha\beta}$ such that $\sum_{y\in U_2} w(y)$ = t(s + 1) – a = sa + [(t-a) + s(t-a)]. The required factorization consists of the valid pair (t - a, s(t - a)) assigned to the edge (u,v) and an $Z_{\alpha\beta}$ -factorization of L', which supplies the remaining weight, a on U_1 and sa on U_2, by means of proposition 2.1. The valid pair (0,0) is assigned to every edge of L\ L', other than (u,v). □

2.2 $S^{\alpha,\beta}$-decompositions of stars

Lemma 2.3 *Let r > 1 be strictly smaller than $\frac{\alpha}{\beta}$. There exists $\Delta_0(\alpha,\beta,r)$ such that the following conditions are sufficient for an $S^{\alpha,\beta}$-decomposition of an n-star, with a sequence $\{h_1, \dots, h_n\}$ of multiplicities on its n edges, of which h_1 is the largest.*

1. $\sum_{i=1}^{n} h_i$ *is divisible by α + β and, if n = 2, also* $\alpha h_1 \equiv \beta h_2$ *(mod* $\alpha^2 - \beta^2$*).*
2. *There exists a sequence of positive real numbers* $\{h'_1, \dots, h'_n\}$*, each larger than* Δ_0*, such that* $h_1' \le r\sum_{i=2}^{n} h'_i$ *and for every i,* $1\le i\le n$*,* $|h_i-h'_i|<\alpha-\beta^2$.

Proof: Let us consider first the case where n = 2: As previously observed, a 2-star with multiplicities h_1 and h_2 is $S^{\alpha,\beta}$-decomposable if and only if there exist non-negative integers a and b such that $h_1 = a\alpha + b\beta$ and $h_2 = b\alpha + a\beta$. If Condition 1 is met then Lemma 2.1 states the existence of integers a', b', k_1 and k_2 such that $h_1 = a'\alpha + b'\beta + k_1(\alpha^2 - \beta^2)$ and $h_2 = b'\alpha + a'\beta + k\_2(\alpha^2 - \beta^2)$. Let $a = a' + k_1\alpha - k_2\beta$ and $b = b' + k_2\alpha - k_1\beta$ to obtain $h_1 = a\alpha + b\beta$ and $h_2 = b\alpha + a\beta$. These equations yield $h_2 = \frac{\beta}{\alpha} h_1 + b\frac{\alpha^2-\beta^2}{\alpha}$ and $h_2 = \frac{\alpha}{\beta} h_1 - a\frac{\alpha^2-\beta^2}{\beta}$. Thus a and b are both non-negative if and only if $\frac{\beta}{\alpha} h_1 \le h_2 \le \frac{\alpha}{\beta} h_1$. The leftmost inequality suffices, given $h_1 \ge h_2$. Let us recall the conditions $|h_i - h'_i| < \alpha^2 - \beta^2$ for i = 1,2 and $h'_1 \le r\, h'_2$ for some r strictly smaller than $\frac{\alpha}{\beta}$. If we select $\Delta_0 > (\alpha^2 - \beta^2)\frac{1+\frac{\alpha}{\beta}}{\frac{\alpha}{\beta}-r}$, then $h'_2 > \Delta_0$ would provide the required inequality.

Assume now n ≥ 3: Define W= $\sum_{i=1}^{n} h_i$ and k= $\frac{W}{\alpha+\beta}$. To obtain the required decomposition, each h_i should be partitioned into a_i summands of size α, and b_i summands of size (thus $h_i = a_i\alpha + b_i\beta$), such that each α-summand can be matched to a β-summand of another h_j to form k copies of $S^{\alpha,\beta}$. Necessary and sufficient conditions for such a perfect matching to exist are:

1. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ = k and
2. $a_i + b_i \le k$, for every i, 1 ≤ i ≤ n

Condition 1 is obvious. Condition 2 is imposed by Hall's condition for the existence of a perfect matching: An α-summand can be matched to any β-summand, unless they are parts of the same h_i. Hall's condition for the set of all α-summands of h_i is hence $a_i \le \sum_{j\ne i}^{n} b_j$ which is equivalent to $a_i + b_i \le k$ (Other sets of summands should not be considered since any β-summand can be matched to at least one of two α-summands of distinct h_is).

Let us define a sequence $\{x_i\}_1^n$ by $a_i = \frac{h_i}{\alpha+\beta} - x_i\beta$, which implies $b_i = \frac{h_i}{\alpha+\beta} + x_i\alpha$.

In terms of these new parameters conditions 1. and 2 can be equivalently restated as:

1. $\sum_{i=1}^{n} x_i = 0$ and
2. $x_i \le \frac{W-2h_i}{\alpha^2-\beta^2}$, for every i, 1 ≤ i ≤ n

Since $a_i\alpha + b_i\beta = (a_i - k\beta)\alpha + (b_i + k\alpha)\beta$, the coefficient a_i can be selected, for every fixed I (assuming h_i is not too small), from an arithmetic progression of difference β, where the smallest element is smaller than β. Similarly, the smallest possible value of b_i is smaller than α. Accordingly, x_i can be selected from an arithmetic progression of difference 1, where the smallest element is smaller than $1 - \frac{h_i}{\alpha(\alpha+\beta)}$ and the largest is larger than $\frac{h_i}{\beta(\alpha+\beta)} - 1$. Combining this last outcome with Condition 2 we get

3. For every i, 1 ≤ i ≤ n, the value of x_i can be selected from an arithmetic progression of difference 1, where the smallest element is smaller than $1 - \frac{h_i}{\alpha(\alpha+\beta)}$ and the largest is larger than $\min\{\frac{h_i}{\beta(\alpha+\beta)}-1, \frac{W-2h_i}{\alpha^2-\beta^2}-1\}$

Regardless of Condition 1, $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n}\frac{h_i}{\alpha+\beta} - \beta\sum_{i=1}^{n} x_i = k - \beta\sum_{i=1}^{n} x_i$ implies that $\beta\sum_{i=1}^{n} x_i$ is an integer. Similarly $\alpha\sum_{i=1}^{n} x_i$ is also an integer and since *gcd*(α,β) = 1, the sum $\sum_{i=1}^{n} x_i$ is an integer. Therefore, instead of directly considering Condition 1,.it suffices to verify that the bounds set in Condition 3 allow a sequence of x_i's, for which $\sum_{i=1}^{n} x_i \le 0$ and one for which $\sum_{i=1}^{n} x_i \ge 0$.

If $h_i \ge \alpha(\alpha + \beta)$ (which is obtained by selecting $\Delta_0 > \alpha(\alpha + \beta) + (\alpha^2 - \beta^2)$, the lower bound $1 - \frac{h_i}{\alpha(\alpha+\beta)}$ is negative for every h_i and hence a sequence for which $\sum_{i=1}^{n} x_i \le 0$ clearly exists.

It remains now to require min{ $\frac{h_i}{\beta(\alpha+\beta)}-1$, $\frac{W-2h_i}{\alpha^2-\beta^2}$ -1} ≥ 0 . Due to the last lower bound for h_i, terms of the form $\frac{h_i}{\beta(\alpha+\beta)}-1$ are positive. The sum of any two terms $\frac{W-2h_i}{\alpha^2-\beta^2}$ -1 + $\frac{W-2h_j}{\alpha^2-\beta^2}$ -1 = $\frac{2}{\alpha^2-\beta^2}\sum_{q\neq i,j} w_q - 2$ is always positive (properly selecting Δ_0 we get $h_i > \alpha^2 - \beta^2$, and this is the only place where N > 3 is indeed essential). Recalling that h_1 is maximum in $\{h_i\}_{i=1}^n$ we should only verify $\frac{W-2h_1}{\alpha^2-\beta^2}$ -1 + $\sum_{i=2}^n(\frac{h_i}{\beta(\alpha+\beta)}-1)$ ≥ 0, which can be straightforwardly restated as: $h_1 \le \frac{\alpha}{\beta}\sum_{i=2}^n h_i - n(\alpha^2-\beta^2)$. Since $|h_i - h'_i| < (\alpha^2 - \beta^2)$ it suffices to require $h_1' + \alpha^2 - \beta^2 \le \frac{\alpha}{\beta}\sum_{i=2}^n (h_i' - (\alpha^2-\beta^2)) - n(\alpha^2-\beta^2)$ or equivalently: $h_1' \le \frac{\alpha}{\beta}\sum_{i=2}^n h_i' - (n+1+\frac{\alpha}{\beta}(n-1))(\alpha^2-\beta^2)$ .The lemma assumes $h_1' \le$ r $\sum_{i=2}^n h_i'$ and hence the required inequality is achieved if $\sum_{i=2}^n h_i'(\frac{\alpha}{\beta}-r) \ge (n+1+\frac{\alpha}{\beta}(n-1))(\alpha^2-\beta^2)$, which holds if for every i ≥ 2, $h_1' \ge \frac{1}{\frac{\alpha}{\beta}-r}(\frac{n+1}{n-1}+\frac{\alpha}{\beta})(\alpha^2-\beta^2)$. The above is reached if we select $\Delta_0 > \frac{1}{\frac{\alpha}{\beta}-r}(2+\frac{\alpha}{\beta})(\alpha^2-\beta^2)$ □

2.3 Constructing $S^{\alpha,\beta}$-decompositions

Definition *Let r and δ be positive real numbers for which: r > 1, 1-δ ≥ δ and r(1-δ) ≥ 1, that is, 0<δ≤min{$\frac{1}{2}, 1-\frac{1}{r}$}.*

1. *A sequence of positive real numbers is (r, δ)-balanced if $\delta \le f_i \le r(1-\delta)$ for every $1 \le i \le n$.*
2. *An (r, δ)-balanced sequence \{$f_1,...,f_n$}, of at least two elements, is (r, δ)-feasible if $f_i \le r\sum_{j\neq i} f_i$ for every $1 \le i \le n$. (It clearly suffices to consider i for which f_i is maximum). A singleton \{f_1} is feasible if $f_1 = 0$. When the context allows `(r, δ)-' is omitted from '(r, δ)-balanced' and `(r, δ)-feasible.'*
3. *Let G=(V,E), be a graph and w : E → R a real valued weight function on its edges. An (r, δ) decomposition of G is a two-variables function f which partitions w(e) on each edge e = (x,y) into two summands $w(e) = f_x(e) + f_y(e)$ such that for every vertex x ∈ V, incident with edges $e_1,...,e_n$, the sequence {$f_x(e_1),...,f_x(e_n)$} is feasible.*
4. *A graph G=(V,E) is (r, δ)-resolvable if for every real valued function w : E → [1,r(1-δ)] there exists an (r, δ)-decomposition of (G,w).*
5. *A graph is randomly resolvable if it is (r, δ)-resolvable for every r > 1 and 0 < δ ≤ min{$\frac{1}{2}, 1-\frac{1}{r}$}.*

Whenever r and δ are used in the context of an (r, δ)-decomposition, we assume r > 1 and 0 < δ ≤ min{$\frac{1}{2}, 1-\frac{1}{r}$}. A weight function w, mentioned in that context, have range [1,r(1-δ)], unless explicitly stated otherwise.

The role of the last definitions lies in the following lemmas, which are the core of our schema for constructing $S^{\alpha,\beta}$-decompositions:

Lemma 2.4 *For every r, 1 < r < $\frac{\alpha}{\beta}$, and δ, 0<δ≤min{$\frac{1}{2}, 1-\frac{1}{r}$}, there exists $\lambda_0(\alpha,\beta,r,\delta)$, such that: If a graph G = (V,E) is (r, δ)-resolvable and w : E → [λ,r(1-δ)λ], for some λ > λ_0, then (G,w) admits an $S^{\alpha,\beta}$-decomposition if and only if*

1. *$\sum_{e\in E} w(e)$ is divisible by α + β*
2. *If G is either a path, or an even circuit, then*

$\alpha\sum_{e \text{ is odd}} w(e) \equiv \beta\sum_{e \text{ is even}} w(e)$ (mod $\alpha^2 - \beta^2$), where 'odd' and `even' refer to the location of the edge e along the path, or even circuit (with an arbitrary first vertex).

Proof: Let $\Delta_0(\alpha,\beta,r)$ be as stated in Lemma 2.3, and $\lambda_0 = \frac{\Delta_0}{\delta}$. Since $\frac{w}{\lambda}$: E → [1,r(1-δ)], there exists an (r,δ)-decomposition f of (G, $\frac{w}{\lambda}$).
The function h' = fλ then associates every vertex x ∈ V, incident with edges {$e_1,...,e_n$}, with a sequence {$h'_x(e_1),...,h'_x(e_n)$} which satisfies:

1. $h'_x(e_i) \ge \Delta_0$ (or $h'_x(e_i) = 0$ if d(x) = 1))
2. $h'_x(e_i) \le$ r $\sum_{j\neq i} h'_x(e_j)$

Also for every edge e = (x,y):

3. $w(e) = h'_x(e) + h'_y(e)$.

Now let g be a $Z_{\alpha,\beta}$-decomposition of (G,w), whose existence is implied by Lemma 2.2. According to Definition:

4. $g_x(e)+g_y(e) \equiv w(e)$(mod α^2-β^2) for every edge e∈E.

For a vertex x ∈ V, incident with edges $e_1,...,e_n$, the sequence {$g_x(e_1),...,g_x(e_n)$} is the weight function on the edges of a $Z_{\alpha,\beta}$-decomposable n-star and as such, complies with the divisibility conditions of Lemma 2.2, which are the same divisibility conditions stated in Lemma 2.3 , for every vertex x of degree 2 or more, and an edge e incident with x. We select from the interval [$h'_x(e) - (\alpha^2 - \beta^2)$, $h'_x(e) + (\alpha^2 - \beta^2)$], an integer $h_x(e)$ congruent to $g_x(e)$ modulo $\alpha^2 - \beta^2$, such that, for every vertex u ∈ U

5. $h_x(e) + h_y(e) = w(e)$.

Let us verify that such a selection is indeed possible: If each $h_x(e)$ is congruent to $g_x(e)$ modulo α^2-β^2 then Condition 4 supplies the necessary congruence of the sum. Considering 3 and the existence of two candidates for $h_x(e)$ in each given interval, one smaller and one larger than $h'_x(e)$, there exists a selection for which Condition 5 indeed holds.

Each pair of sequences h = {$h_x(e_1),...,h_x(e_n)$} and h' = {$h'_x(e_1),...,h'_x(e_n)$} complies with the conditions of Lemma 2.3 and therefore the star centered at each vertex x ∈ V admits an $S^{\alpha,\beta}$-decomposition with multiplicities {$h_x(e_1),...,h_x(e_n)$}. Together with Condition 5 we meet the definition for an $S^{\alpha,\beta}$-decomposition of (G,w). □

Lemma 2.5 *Let α be larger than β and let ε be any constant for which 0 < ε < 1 - $\frac{\beta}{\alpha}$, then there exists $\lambda_0(\alpha,\beta,\varepsilon)$ such that for every λ > λ_0 :*

If G is randomly-resolvable and w : E → [λ, $\frac{\alpha}{\beta}$ (1-ε)λ] , then (G,w) admits an $S^{\alpha,\beta}$-decomposition if and only if

1. *$\sum_{e\in E} w(e)$ is divisible by α + β*
2. *If G is either a path, or an even circuit, then*
   $\alpha\sum_{e \text{ is odd}} w(e) \equiv \beta\sum_{e \text{ is even}} w(e)$ (mod $\alpha^2 - \beta^2$).

Proof: Select any r such that $\frac{\alpha}{\beta}(1-\varepsilon)$ < r < $\frac{\alpha}{\beta}$ and δ, $0 < \delta \le \min\{\frac{1}{2}, 1-\frac{1}{r}\}$, such that $\frac{\alpha}{\beta}(1-\varepsilon)$ < r(1 - δ), then apply Lemma 2.4 . □

2.4 Conditions for (r.δ)-resolvability

The following ad hoc notation helps us dealing with trees, where edges, rather than vertices, are at the center of attention:

1. An edge , incident with a vertex of degree 1 , is *a leaf* (or *a leaf-edge*, where confusion with conventional vertex notation might be caused).
2. A tree can be *rooted* at any selected leaf-edge, that then becomes the *root* of the tree.
3. Let s = (x,y) be the root of a tree. An edge s_1 = (y,z) ≠ s, is a *lower-neighbor* of s, and s is the *parent* of s_1.
4. Let s = (x,y) be the root-edge of a tree T and e = (y,z) a lower-neighbor of s, then e is the root of the connected component which includes that edge, obtained by deleting all other edges incident with y. This component is the *subtree of T, rooted at* e.
5. Recursively, once a root is selected to a tree T, every edge e becomes the root of a certain subtree; it has a unique parent (unless it is the root of T) and possibly some lower-neighbors.
6. A rooted tree T=(V,E) is *even*, or *odd* according to the parity of |E|. The parity of an edge e ∈ E is that of the subtree rooted at e.
7. A rooted tree is *Regular* if no odd edge has an odd lower neighbor and every even edge has exactly one odd lower neighbor. If L is not regular then it is *irregular*.
8. A tree T, rooted at s = (x,y), d(x) = 1 is *almost randomly resolvable* (ARR) if for every r, 1 < r < $\frac{\alpha}{\beta}$, and δ, $0 < \delta \le \min\{\frac{1}{2}, 1-\frac{1}{r}\}$, and w : E → [1,r(1-δ)], there exists t, δ ≤ t ≤ r(1 - δ), such that (T,w') admits an (r,δ)-decomposition f, where w'(s) = $f_y(s)$ = w(s) - t and w'(e) = w(e) for any other edge e. We say that f is an (r,δ)-decomposition of (T,w) with *residual weight* t on the root s. If f is to be expanded to an (r,δ)-decomposition of a larger graph where d(x) > 1 then $f_x(s)$ = t.
9. A tree T, rooted at s = (x,y), d(x) = 1 is (r,δ)-*strongly resolvable* if for every w such that δ ≤ w(s) ≤ r(1 - δ) and w(e) ∈ [1, r(1 - δ)] , for any other edge e, (T,w) is (r,δ)-decomposable. If T is (r,δ)-strongly resolvable, it is clearly (r,δ)-resolvable.
10. A tree is *strongly randomly resolvable* (SRR) if for every r, 1 < r < $\frac{\alpha}{\beta}$, and δ, $0 < \delta \le \min\{\frac{1}{2}, 1-\frac{1}{r}\}$, it is (r,δ)-strongly resolvable. If T is SRR it is clearly randomly resolvable.

Proposition 2.3 *If $\{f_1,f_2\}$ is (r.δ)-feasible, $f_1 + f_2 \ge 1-\delta$ and $\{f_1,f_2,...,f_n\}$ is(r.δ)-balanced, then $\{f_1,f_2,...,f_n\}$ is (r.\delta)-feasible.*

Proof: If one of $\{f_1,f_2\}$, say, f_1 is maximum in $\{f_1,f_2,...,f_n\}$ then $f_1 \le rf_2 \le r\sum_{i=2}^{n} f_i$. Otherwise $f_{i\ne 1,2} \le r(1-\delta) \le r(f_1 + f_2) \le r\sum_{j\ne i} f_i$. □

Proposition 2.4 *For every r, 1 < r < $\frac{\alpha}{\beta}$, and δ, $0 < \delta \le \min\{\frac{1}{2}, 1-\frac{1}{r}\}$, if $\{f_1,f_2,...,f_n\}$is (r.δ)-balanced then there exists a positive real number z, δ ≤ z ≤ 1 - δ such that $\{f_1,f_2,...,f_n,z\}$is (r.δ)-feasible.*

Proof: Select x = $\max\{\delta, \frac{1}{r} f_m\}$, where $f_m = \max\{f_1,f_2,...,f_n\}$. Now $f_m \le rx \le r(x + \sum_{i=m}^{n} f_m$. □

Lemma 2.6 *Every tree T is ARR.*

Proof: If T consists of a single edge s then the residual weight is t = w(s). If not, apply induction to the trees $T_1,...T_n$ rooted at the lower-neighbors of s, to get a feasible sequence $\{t_1,...t_n, z\}$, where t_i is the residual weight on the root of T_i and z is obtained by Proposition 2.4 . Since z ≤ 1 - δ, the residual weight t = w(s) - z is indeed in the proper range. □

Lemma 2.7 *A connected graph G which is not a tree is randomly resolvable.*

Proof: Let G = (V,E) and w : E → [1,r(1-δ)]. Let $E_c \subset E$ be the set of all edges e such that, either e belongs to a circuit, or each of the two connected components, obtained when e is removed, contains a circuit. Let V_c be the set of vertices of the subgraph spanned by E_c. The edges in E \ E_c are partitioned into trees, each rooted at an edge s = (x,y) with exactly one vertex x in V_c. Applying Lemma 2.6, we decompose T with a residual weight $f_x(s)$ = t on s. For every $e=(u,v)\in E_c$ we define $f_u(e) = f_v(e) = \frac{w(e)}{2}$. If a vertex v is incident with an edge of E_c then it is incident with at least two such edges e_1,e_2. Since $f_v(e_1)+f_v(e_2) = \frac{1}{2}(w(e_1)+w(e_2)) \ge 1$, the sequence $\{f_v(e)|$ e is incident with v} is feasible by Proposition 2.3, and f is indeed an (r,δ)-decomposition of (G,w). □

Lemma 2.8 *If for some edge e of a tree T, the subtree rooted at e is (r,δ)-strongly resolvable then T is (r,δ)-strongly resolvable. Consequently, if the subtree is SRR then T is SRR.*

Proof: Let T be a tree, rooted at s = (x,y) and let s' = (y,z) be a lower-neighbor of s, such that the subtree T', rooted at s' is (r,δ)-strongly resolvable. Let w(s) be in [δ, r(1 - δ] and w(e) ∈ [1,r(1-δ)] for any other edge e. Let $t_1,...t_n$ be the residual weights on all lower-neighbors of s, other than s', obtained by Lemma 2.6. Apply Proposition 2.4, to find z which makes the sequence $\{t_1,...,t_n,w(s),z\}$ feasible. Since T' is (r,δ)-strongly resolvable it can be decomposed with residual weight z on s' and hence T is (r,δ)-strongly resolvable. The assertion of the Lemma follows by induction □

Since every tree is ARR, we can relate to the residual weight on the root of any tree:

Lemma 2.9 *Given a rooted tree, T, the parameters r and δ and a weight function w in the proper range, the set of the residual weights on the root s, over all (r,δ)-decompositions of T with residual weight, is an interval.*

Proof: Considering w(e), $f_x(e)$ and $f_y(e)$ over all edges e = (x,y) ∈ E as variables. The definition of an (r,δ) decompositions (with or without residual weight), forms a convex polytope. The assertion of this lemma immediately follows. □

Lemma 2.10 *The residual weight t on the root s of a regular tree T=(V,E), can be freely selected from an interval which:*

- *Contains [0,kδ] if T is even on 2k edges, or*
- *Includes a and b, such that a ≥ 1-δ and b ≥ r(1- kδ) if T is odd on 2k-1 edges*

As far as $1 \le k < \frac{1}{\delta}$

Proof: If T consists of a single edge then T is odd, k = 1 and a = b = t = w(s) is indeed in the right range. We proceed by induction on the height of T:

Let s = (x,y), d(x) = 1, be the root of an odd tree T on 2k+1 edges and let $T_1,...,T_n$ be the (all even) subtrees rooted at the lower neighbors $e_1,...,e_n$ of s. Let each T_i contain $2k_i$ edges, $\sum_{i=1}^{n} k_i$ = k. By the induction hypothesis, each residual weight $t_i = f_y(e_i)$ can be aimed at any value in the interval $[\delta,k_i\delta]$ (a smaller residual weight cannot be used). If we select $t_i = \delta$ for every i then the sequence can be made feasible by an additional δ, that is $f_y(s) = \delta$, implying a residual weight a = w(s) - δ , which is at least 1-δ. On the other hand if for every i, $t_i = k_i\delta$ then we can set $f_y(s) = r \sum_{i=1}^{n} k_i\delta$ = rkδ. The residual weight on s is then b = w(s) - rkδ, which is at most r(1 - (k + 1)δ). The odd case is completed, considering Lemma 2.9 .

Let T be even on 2k edges. Again $T_1,...,T_n$ are the subtrees, rooted at the lower neighbors $e_1,...,e_n$ of s = (x,y). This time we assume that T_1 is odd on $2k_1$-1 edges and for i ≥ 2 (if there is any) each T_i is even on $2k_i$ edges, $\sum_{i=1}^{n} k_i$ = k. Decompositions of $T_1,...,T_n$ again yield residual weight $t_i = f_y(e_i)$ on each edge $e_2,...,e_n$, which can be selected from $[\delta,k_i\delta]$. T_1 is odd and hence the interval from which the residual weight $t_1 = f_y(e_1)$ can be chosen, contains a and b, a ≥ 1 - δ and b ≤ $r(1 - k_1\delta)$. By selecting $t_1 \geq 1 - \delta$, and $f_y(s)$ = w(s), the sequence $\{t_1,...,t_n,f_y(s)\}$ becomes feasible due to lemma 2.3 and the residual weight on s is w(s) – $f_y(s)$ = 0.

We now choose $f_y(s)$ = 1 - kδ and show that $t_1,...,t_n$ can be selected to make $\{t_1,...,t_n,f_y(s)\}$ feasible. We start with $t_1 = b \leq r(1 - k_1\delta)$ and $t_i = k_i\delta$ for i ≥ 2. We obtain $t_1 \leq r(\sum_{i=2}^{n} t_i + f_y(s))$, which provides feasibility in the case where t_1 is the maximal term. If another term t_j is the maximum, we reset $t_1 = t_j$, this is possible because the interval from which t_1 is selected contains a and b where a ≥ 1-δ ≥ t_j > b. Now there are two equal maximal terms and hence the sequence is feasible. Since w(s) ≥ 1 the residual weight s(w) – $f_y(s)$ can be made at least as large as 1-kδ. The induction is completed by lemma 2.9.

Lemma 2.11 *A regular tree T=(V,E) with |E| ≥ $\frac{2}{\delta}$ edges is (r,δ)-strongly resolvable.*

Proof: Let T = (V,E), rooted at s be a tree with |E| ≥ $\frac{2}{\delta}$, such that each of the trees $T_1,...,T_n$, rooted at the lower-neighbors $e_1,...,e_n$ of s has less than $\frac{2}{\delta}$ edges, and, therefore can be subject to Lemma 2.10. Take now the induction step in the proof of Lemma 2.10 one step farther. In both parts of the induction - T even, or odd, we now obtain $\sum_{i=1}^{n} k_i = k = \frac{1}{\delta}$. Consequently kδ > 1, which allows any residual weight from 0 to r(1 - δ) - δ. The tree T is then (r,δ)-strongly resolvable and the same holds for any regular tree with that many edges by Lemma 2.8 . □

Lemma 2 12 *A graph is randomly resolvable, unless it is an odd regular tree.*

Proof: In light of Lemma 2.7 we can restrict the discussion to trees. Residual weight 0 on the root means an (r,δ)-decomposition, hence Lemma 2.10 states that every even regular tree is randomly resolvable. We will now prove that an irregular tree is SRR. Let s be the root of a minimal irregular tree T=(V,E). By minimality of T the subtrees $T_1,...,T_n$ rooted at the lower neighbors $e_1,...e_n$ of s are regular and at least two of them, say, T_1 and T_2 are odd. Lemma 2.10 allows decompositions of $T_1,...,T_n$ with residual weight of at least 1 - δ on both e_1 and e_2. The sequence consisting of the residual weights on $e_1,e_2,...,e_n$ and of w(s) is feasible, by Lemma 2.3, regardless of any term (particularly w(s)), but the first two. T is thus proved SRR and, by Lemma 2.8, so is any tree which contains a minimal irregular tree, namely every irregular tree. □

2.5 The main result

Theorem 2.1 *Let α be larger than β and let ε be any constant $0 < \varepsilon < 1 - \frac{\beta}{\alpha}$, then there exist $\lambda_0(\alpha,\beta,\varepsilon)$ and $M_0(\alpha,\beta,\varepsilon)$ such that for every $\lambda > \lambda_0$,*
If G=(V,E) is

- *Any connected graph, other than an odd regular tree, or*
- *An odd regular tree where $|E| \geq M_0$,*
  And
- *$w : E \to [\lambda, \frac{\alpha}{\beta}(1 - \varepsilon)\lambda]$.*

then (G,w) admits an $S^{\alpha,\beta}$-decomposition if and only if

3. *$\sum_{e\in E} w(e)$ is divisible by α + β*
4. *If G is either a path, or an even circuit, then*
   $\alpha \sum_{e \text{ is odd}} w(e) \equiv \beta \sum_{e \text{ is even}} w(e) \pmod{\alpha^2 - \beta^2}$.

Proof: If G is not an odd regular tree, combine Lemmas 2.5 and 2.12 . If G is an odd regular tree, follow the proof of Lemma 2.5 to obtain r,δ, and λ_0. Finally select $M_0 = \frac{2}{\delta}$ and apply Lemma 2.11 . □

2.6 Optimizing the parameters

In this subsection we compute explicit bounds for λ_0, and M_0, once α, β and ε are given, and reach the following conclusion:

- The parameter λ_0 of Theorems 2.1 is roughly $(4 + 8\frac{\beta}{\alpha})(\alpha^2 - \beta^2)/\varepsilon^2$, if ε is small and approximately $(5 \pm 1)\alpha^2$ if ε > 3/4 (and hence α > 4β).
- The parameter M_0 that sets the minimum number of edges when the input graph is an odd regular tree is of order $\frac{4}{\varepsilon}$, if ε is small. No such bound is required when ε ≥ 1/2(possible if α > 2β).
- When ε <1/2, the bound M_0 (for an odd regular tree) can be pushed down as closely as wished to 2/ε, but then λ0 is increased proportionally to $\frac{1}{\varepsilon - 2/M_0}$.

Here is the actual computation :

Recalling the proofs of Lemmas 2.3 and 2.5, it appears that $\lambda_0 \geq \frac{\Delta_0}{\delta}$ suffices, where the dominant bounds for Δ_0 are as following

$$\Delta_0 \geq \frac{1}{\frac{\alpha}{\beta} - r}(2 + \frac{\alpha}{\beta})(\alpha^2 - \beta^2) . \quad (1)$$

$$\Delta_0 \geq \alpha(\alpha + \beta)\, \alpha^2 - \beta^2 \quad (2)$$

We will show in the sequel that, unless both $\frac{\alpha}{\beta}$ and ε are rather large, Eq. 1 is a stronger requirement than Eq. 2. Let us consider first Eq. 1 only. Minimizing λ in that case is equivalent to maximizing $\delta(\frac{\alpha}{\beta} - r)$, subject to the constraints $\frac{\alpha}{\beta}(1-\varepsilon) \leq r(1 - \delta)$ and $0 < \delta < \frac{1}{2}$, posed in the proof of Lemma 2.5 and Definition . The additional condition r(1-δ)≥1, is implicit in $\frac{\alpha}{\beta}(1 - \varepsilon) \leq r(1 - \delta)$. For the sake of convenience we define $\rho = \frac{\beta}{\alpha}$, q = rρ and θ = 1 - δ, and maximize $\frac{1}{\rho}(1 - \theta)(1 - q)$, subject to qθ ≥ 1 - ε and $1 > \theta \geq \frac{1}{2}$. If $\varepsilon \leq \frac{3}{4}$, the maximum is reached at $q = \theta = \sqrt{1-\varepsilon}$ and its value is $\frac{1}{\rho}(1 - \sqrt{1-\varepsilon})^2$. Accordingly, $\lambda_0 = (1 + 2\frac{\beta}{\alpha})(\alpha^2 - \beta^2)/(1 - \sqrt{1-\varepsilon})^2$ would suffice for Theorem2.1 . If ε is small then this value is roughly $(4 + 8\frac{\beta}{\alpha})(\alpha^2 - \beta^2)/\varepsilon^2$, as claimed above.

At that point $\delta = (1 - \sqrt{1-\varepsilon})$, which implies $\Delta_0 = (1 + 2\frac{\beta}{\alpha})(\alpha^2 - \beta^2)/(1 - \sqrt{1-\varepsilon})$. Thus Eq. 2 becomes relevant if $\alpha(\alpha + \beta)\ \alpha^2 - \beta^2 \geq (1 + 2\frac{\beta}{\alpha})(\alpha^2 - \beta^2) / (1 - \sqrt{1-\varepsilon})$. That is (after some algebraic manipulation) , when $\sqrt{1-\varepsilon} \leq \frac{1-2\rho+2\rho^2}{2-\rho}$, where ρ stands, again, for $\frac{\beta}{\alpha}$. This clearly implies $\varepsilon > \frac{3}{4}$ and, since our theorems assume $1 - \varepsilon > \frac{\beta}{\alpha}$, also $\frac{\alpha}{\beta} > 4$.

For that range of the parameters we go back to the maximization schema we derived from Eq. 1. Now the maximum is obtained at $q = \delta = \frac{1}{2}$ and the bound becomes $\lambda_0 > (1 + 2\frac{\beta}{\alpha})(\alpha^2 - \beta^2)/(\varepsilon - \frac{1}{2})$. For that value of δ we also obtain $\lambda_0 \geq 2(\alpha(\alpha + \beta) + \alpha^2 - \beta^2)$ from Eq. 2. This way or the other, since $\varepsilon > 3/4$ and $\alpha > 4\beta$ the bound for λ_0, in that case is between $4\alpha^2$ and $6\alpha^2$.

When the graph at hand is an odd regular tree, there is also a lower bound of $M_0 = \frac{2}{\delta}$ on the number of edge (Theorem 2.1 and its proof). This bound equals $2/(1 - \sqrt{1-\varepsilon})$, if $\varepsilon < 3/4$, or roughly $\frac{4}{\varepsilon}$ for small ε. No M_0 (well, at least 2) is required if $\varepsilon > 1/2$ (Although for $\varepsilon = 1/2$, our computation so far yields $M_0 \approx 6.8$, there are only three odd regular trees with three or five edges. Of these, two are paths. For paths, in general, smaller M_0 suffices because the constraints obtained from the analysis of n-stars, $n \geq 3$ does not apply. We omit the explicit discussion of the remaining tree of five edges).

We recall the analysis of Eq. 1 once more, this time with the minimization of M_0 in mind, for the case of an odd regular tree, and $\varepsilon < 1/2$. This is clearly equivalent to maximizing δ. The given constraints allows to set δ as close to ε, as wished, that is M_0 as close to $\frac{2}{\varepsilon}$ as wished. The price is increasing λ_0, proportionally to $\frac{1}{\varepsilon-\delta}$.

3. THE INTRACTABLE CASES

The upper bound $\frac{\alpha}{\beta} - \varepsilon$ on the ratio between the smallest and the largest multiplicities, presented in the previous section, is clearly tight. Take for example a multigraph M on an underlying simple path with 2n+1 edges, where the multiplicity alternates between $\alpha\lambda$ and $\beta\lambda$ on the first 2n edges and its value on the last edge is arbitrarily set to comply with the necessary divisibility condition. M does not admit an $S^{\alpha,\beta}$-decomposition, no matter how large λ and n are.

A similar example, where the ratio between the alternating multiplicities is slightly less than $\frac{\alpha}{\beta}$ shows that the bounds we set for λ_0 and M_0 are also rather tight.

Although these bounds are clearly not necessary, there is not much hope (as long as P versus NP is not settled) to characterize decomposable multigraphs which do not obey them: In this section we prove that $S^{\alpha,\beta}$-decomposition becomes NP-Complete even when the multiplicities are arbitrarily large if the smallest multiplicity is larger by a constant than $\frac{\beta}{\alpha}$ times the largest one.

On the other direction we show NP-completeness of $S^{\alpha,\beta}$-decomposition for any pair (α,β) when small multiplicity is allowed, even if the multiplicity range is restricted to a single value. However, we are able to do that for a specific constant multiplicity (of size αβ to be precise) and we do not know what the complexity status is when larger (yet not as large as λ_0) multiplicities are allowed within a bounded ratio range. We state now these two results:

Theorem 3.1 *$S^{\alpha,\beta}$-αβ decomposition is NPC.*

Theorem 3.2 *For any positive constant integer $\lambda > \alpha$, $S^{\alpha,\beta}$-decomposition is NPC, even if the input's multiplicity function is bounded to the interval $[(\lambda + 1)\beta, \lambda\alpha]$.*

Proof: The proofs of both theorems are based on the same underlying idea: We construct a multigraph T with two special vertices of degree1 each, u incident with an edge e_u and v incident with e_v, which satisfies the following:

- There are exactly two $S^{\alpha,\beta}$-decomposable subgraphs of T which achieve the full multiplicity of T on every edge except, maybe, on e_u and on e_v.
- One of these subgraphs has the same multiplicity as T on e_u and its multiplicity on e_v is smaller by αβ than that of T, and the same holds for the other one with the roles of e_u and e_v switched.

We start with an input graph G=(V,E) for $S_{\alpha+\beta}$-decomposition (known to be NPC), and replace each edge (u,v) of G by a copy of T (with u and v as in the definition of T), to obtain an input multigraph G' for $S^{\alpha,\beta}$-decomposition

After removing one of the decomposable subgraphs of each copy of T, what remains of G' is an arbitrary decomposition of G into stars, with a constant multiplicity αβ. Since *gcd*(α,β) = 1, the only partitions of αβ into α and β summands consist of α summands of size β each, or vise versa. A star S_n is then $S^{\alpha,\beta}$-αβ decomposable, if and only if $n = k(\alpha+\beta)$ for some integer k (in which case the decomposition consists of kβ edges decomposed into α summands of size β and kα edges decomposed into summands of size α).

Accordingly, an $S^{\alpha,\beta}$-αβ decomposition of G' can be completed if and only if the remaining stars form an $S_{\alpha+\beta}$-decomposition of G.

To complete the proofs of Theorems 3.1 and 3.2 a multigraph T should be constructed, subject to the constraints implied by each theorem on the multiplicity function.

3.1 Constructing T with constant multiplicity αβ

The α or β summands into which the multiplicity of each edge is partitioned in an $S^{\alpha,\beta}$-αβ are divided between the two endvertices, such that around every vertex the number of α and β summands is the same, so they can be matched to form copies of $S^{\alpha,\beta}$. It comes handy to count summands of one size as positive and the others as negative.

Accordingly, an $S^{\alpha,\beta}$-αβ decomposition of a graph G=(V,E) is equivalent to an assignment $f_u(e)$, $f_v(e)$ to every edge $e = (u,v) \in E$, such that either $f_u(e)$ and $f_v(e)$ are non-negative integers and $f_u(e) + f_v(e) = \alpha$, or $f_u(e)$ and $f_v(e)$ are non-positive integers and $f_u(e) + f_v(e) = -\beta$; and for every vertex $v \in V$ the sum of $f_v(e)$ over all edges e incident with v, equals 0. We refer to these two types of edges as α-edges and β-edges,respectively.

Proposition 3.*1* *Let x be a vertex of a graph H, shared by two or more subgraphs, which are otherwise vertex disjoint. Let one of these subgraphs S, be a star on m+1 edges, centered at a vertex c which is not x (so x is a leaf of S). For every integer t, $-\beta < t < \alpha$, there exists m, $1 \leq m \leq \alpha + \beta - 1$, such that in any $S^{\alpha,\beta}$-αβ decomposition of H, $f_x(c,x) = t$. We refer to the vertex x of S as the* connector *of a t-*enforcing star.

Proof: Let us try to $S^{\alpha,\beta}$-$\alpha\beta$ decompose H. The condition $\sum f_c(e) = 0$ over the edges of S implies $f_c(c,x) = (m - a)\beta - a\alpha$ where a and m-a are the numbers of α-edges and β-edges among the other m edges of S. When letting the integer a run from 0 to m, the value of $f_c(c,x)$ can be set to any $m\beta$(modulo($\alpha + \beta$)) integer between $-m\beta$ and $m\beta$. Yet, by definition, $f_v(e)$ is always in the interval $[-\beta, \alpha]$ and for any m, other than $m = \alpha + \beta - 1$, exactly one representative of the relevant residue class resides in that interval. In the other hand, since β and $\alpha+\beta$ are relatively prime, the right selection of m would set $f_c(c,x)$ to any target in $[-\beta, \alpha]$. Selecting m such that $f_c(c,x) = \alpha - t$ if t is positive, or $f_c(c,x) = -\beta - t$ if t is negative ($m = \alpha + \beta - 1$, for either one if t = 0), would imply $f_x(c,x) = t$. □

The graph T consists of:

- A β-enforcing star with connector x
- An "α enforcing gadget" made of α copies of (a-1)-enforcing star with a common connector y and otherwise disjoint,
- A three edges path: $e_u = (u,x)$, (x,y), $(y,v) = e_v$.

When $S^{\alpha,\beta}$-$\alpha\beta$ decomposing (a maximal subgraph of) T, Proposition 3.1 implies $f_x(u,x) + f_x(x,y) = -\beta$ and $f_y(x,y) + f_y(y,v) = \alpha$. If (x,y) is an α-edge then (u,x) is necessarily a β-edge and $f_u(u,x) = 0$, $f_x(u,x) = -\beta$, $f_x(x,y) = 0$, $f_y(x,y) = \alpha$, and $f_y(y,v) = 0$, which means that the edge e_v remains untouched. Similarly, if (x,y) is a β-edge then T is fully decomposed except for e_u which remains untouched. T indeed complies with the requirements.

3.2 Constructing T with multiplicity range $[(\lambda+1)\beta, \lambda\alpha]$

For an integer k, $1 \le k \le \alpha\beta$, let us define a k-couple to be a multigraph on an underlying two edge path, (x,c),(c,d) with multiplicity w((x,c)) = $\lambda\beta + k$ and w((c,d)) = $\lambda\alpha$. As far as $S^{\alpha,\beta}$-$\alpha\beta$ decomposability is the issue, a k-couple, sharing its *connector* x with the rest of a bigger multigraph, behaves as an edge of multiplicity k: It takes λ copies of $S^{\alpha,\beta}$ to saturate the multiplicity of (c,d) and the rest of the graph "sees" the edge (x,c) with the residual multiplicity k.

The graph T consists of the following:

1. A path $e_u = (u,x),(x,z),(z,y),(y,v) = e_v$ with the corresponding multiplicities n_1,n_2,n_3,n_4, where all four are integer products of $\alpha\beta$, taken from the permissible interval (here is where $\lambda > \alpha$ is essential).
2. A set of $\frac{n_2 + n_3 - \alpha\beta}{\beta}$ α-couples, sharing z as a common connector (and otherwise disjoint)
3. A set of $\frac{n_1}{\beta}$ α-couples, sharing x as a common connector.
4. A set of $\frac{n_4}{\alpha}$ β-couples, sharing y as a common connector.

When decomposing T, after all α-couples of Condition 2 are matched to β-summands on (x,z) and (z,y), the sum of the residual multiplicities remaining on these two edges is $\alpha\beta$. As previously observed, this amount is fully partitioned into α-summands, or fully into β-summands. Assume the first option. In that case, none of the α-couples of Condition 3 can be matched to (x,y), so they are all matched onto e_u and that way fully saturate its multiplicity. A simple computation shows that a residual multiplicity $\alpha\beta$ then remains on e_v. Similarly, if the second option is considered , T is fully saturated, except for a residual multiplicity $\alpha\beta$ on e_u. □

REFERENCES

[1] K. Bry's and Z. Lonc. **A complete solution of a Holyer problem.** *4th Twente Workshop on graph and combinatorial optimization, University of Twente, Enschede, The Netherlands, June 1995.*

[2] J. Bosak. **Decompositions of Graphs**. *Kluwer Academic Publishers, 1990.*

[3] J. C. Bermond and D. Sotteau. **Graph decomposition and G-designs.** *Proc. of the 5th British combinatorial conference. Aberdeen, pages 53-72, 1975.*

[4] D. Dor and M. Tarsi. **Graph decomposition is *NP*-complete: proof of Holyer's conjecture**. SIAM *J. Comput., 26:1166-1187, 1997.*

[5] I. Holyer. **The *NP*-completeness of some edge partition problems.** SIAM *J. Compute., 10:713-717, 1981.*

[6] M. Priesler and M. Tarsi. **On some multigraph decomposition problems and their computational complexity**. *Discrete Mathematics, 281:247-254, 2004.*

[7] M. Priesler and M. Tarsi. **Multigraph decomposition into stars and into multistars.** *Discrete Mathematics, 296:235-244, 2005.*

[8] J. Steiner. **Combinatordche aufgabe**. *Z. Reine Angew Math, 45:181-182, 1953.*

[9] R. M. Wilson. **Decomposition of a complete graph into subgraphs isomorphic to a given graph.** *Proc. 5th British combinatorial conference. Aberdeen 1975, Utilitas Math, Winnipeg, pages 647-659, 1976.*