Are We Meeting Pedagogic Requirements?—The Quadratic Equation

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ABSTRACT

Hendel [8] recently proposed four pillars of good pedagogy: executive function, goal-setting, attribution theory and self-efficacy. These pillars are consistent with and supplement the pedagogical hierarchies [1,4,5,16,29,30,31]. These pillars also supplement the National Council of Teachers of Mathematics [NCTM] Process Standards [13,23] as well as the Standards of Mathematical Practice (SMP) of the Common Core State Standards of Mathematics(CCSSM) [11]. A natural follow-up question is whether, and how, current and past textbooks are meeting these requirements. This paper addresses this follow-up question by studying five pre-2000 textbooks [2,6,10,26,27] and three post-2000 books [12,17,28]. For purposes of specificity, the paper exclusively focuses on the treatment of the quadratic function/equation. Using the four pillars, the following questions are asked: What would executive function require for teaching the quadratic function/equation? What does the theory of goal-setting tell us about teaching the quadratic function/equation? What does attribution theory require? The paper's main conclusions are that: i) some pre-2000 textbooks are already meeting the new standards; ii) no single textbook meets all requirements; iii) the requirements of pedagogic excellence—of Hendel, the Process Standards or the SMP—should be met by placing a primary focus on verbal problems. The paper also addresses operational concerns and shows how both operational and pedagogic concerns can be met simultaneously.

Keywords: Executive Function, Goal Setting, Attribution Theory, Self-Efficacy, Quadratic, Common Core State Standards of Mathematical Practice, NCTM Process Standards

1. PEDAGOGIC CHALLENGE

In a recent book [8], Hendel proposes four attributes or pillars that every good pedagogy must have:

- Executive Function: Using multiple modalities of presentation and multiple-parameter explanations [7].
- Goal-Setting Theory: The breakup of complex tasks into a sequence of simpler tasks, each clear, well-defined, specific, challenging, and achievable short term [14,15].
- Attribution Theory: Students must perceive that success is dependent on their own efforts [32].
- Self-Efficacy: Students must believe that with their current skill sets they can achieve the desired course goals [3].

Hendel shows these four pillars consistent with, and supplementing, other definitions of pedagogic challenge such as those of Bloom [4], Anderson [1], Van-Hiele (for geometry) [29], Gagne [5], Marzano [16] and Webb [30,31]. These four pillars also incorporate several decades of research on goal-setting theory [14,15] as well as the importance of student self-efficacy for educational success [3].

The requirements of pedagogic excellence have also been addressed on a national level by the Process Standards of the National Council of Teachers of Mathematics (NCTM) [13,23], as well as the Standards of Mathematical Practice (SMP) proposed by the Common Core State Standards for Mathematics (CCSSM) [11]. A brief history and summary of the Process Standards as well as the CCSSM are now presented.

Throughout the eighties of the last century educators became aware that U.S. students were underachieving [18,19]. The NCTM formed several task-groups to obtain consensus on standards and in 1989 published the curriculum and evaluation standards for school mathematics [20]. Throughout the nineties, these standards evolved to include teacher professional standards and assessment standards [21,22].

This decade of publications on standards culminated in the NCTM publishing the Process Standards in 2000 [23]. The Process Standards call for every student to have the capacity to master and integrate the skills of a) problem solving, b) reasoning and proof, c) communication, d) connections and e) representations. The NCTM identifies the requirements of problem solving as "the process of applying a variety of appropriate strategies based on information provided, referenced, recalled, or developed. Students require frequent opportunities to formulate, grapple with, and solve complex problems that involve a significant amount of effort." However, already in 1989, the NCTM declared that problem-solving should not *follow* the acquisition of skills, but rather should be used to *develop* and *acquire* them.

The goals of the CCSSM were similar to those of NCTM: creation of a uniform curriculum for mathematics across all states so that high-school graduates were prepared for entering two-year colleges, universities and the workforce. The CCSSM, first published in 2010, evolved from over a dozen years of groups, task-forces and reports [11]. The CCSSM identifies eight SMP: i) problem solving, ii) abstract and quantitative reasoning, iii) argumentation and critique, iv) mathematical modeling, v) using appropriate tools, vi) precision, vii) use of structure, and viii) regularity in repeated reasoning.

As can be seen, problem solving is very high on this list. Students who frequently practice and master a variety of problem solving techniques have met most of the SMPs, i), iv), v) and vii). They also fulfill process standards a) and e). Furthermore, if the solution requires a written end-product, these students meet the important process standard, c). Hendel also prefers *verbal* problem solving since executive function is required to integrate the verbal and symbolic skills [7,8].

The purpose of this paper is to evaluate current and past textbooks using the four pillars, the Process Standards and the SMP. For purposes of specificity, one topic, the quadratic equation/function is exclusively focused on.

One goal of this study is to show that the four pillars are easier to use then the Process Standards or SMP; the four pillars are more specific and easier to identify. Furthermore, as will be seen, the four pillars will sometimes require things not explicitly required by the Process Standards and the SMP.

A brief outline of this paper is as follows:

Section 2 applies the four pillars to the quadratic equation/function. The section asks, "What do executive function, goal-setting and attribution theory require to teach the quadratic equation/function?" Five critical areas are identified.

Then, in Section 3, five pre-2000 texts and three post-2000 texts are reviewed for how they approach teaching the quadratic equation/function. A significant finding is that despite the standards, books do not cover and use verbal problems fully. Section 3 presents the paper's main conclusion, that a primary emphasis on a diverse set of verbal problem types can meet the requirements of the four pillars as well as the requirements of the Process Standards and the SMP. Section 4 continues the work of Section 3 by applying goal-setting to the study of the quadratic equation/function. Although goal-setting is one of the four pillars, it is not *explicitly* addressed, neither by the Process Standards nor by the SMP. The section further explores how verbal problems can be used to enhance student communication skills (Process Standard c).

2. THE QUADRATIC EQUATION/ FUNCTION

Although, the topic "quadratic equation/function" may sound simple, this section shows that this topic has a great deal of richness, breadth and application. The purpose of this section is to apply the four pillars of good pedagogy mentioned in Section 1 to the quadratic equation/function. There are five critical quadratic equation/function areas:

2.1. Verbal Problems

The first pillar of good pedagogy is executive function. Four aspects of executive function are explored.

The first example of executive function is *verbal* problems. What types of verbal problems are modeled by quadratic equations? A review of many textbooks, [2,6,10,12,17,26,27,28] identifies seven main classes of quadratic-equation verbal problems. Table 1, summarizes these major areas of verbal quadratic problems.

Two omissions from Table 1 should be mentioned: i) Almost all books present verbal problems where some real-world phenomena (other than the seven areas listed in Table 1) is *modeled* by a quadratic equation; ii) certain verbal problems, for example, quadratic regression [28] require prerequisites from other fields (such as scatter plots and regression).

2.2. The Rule of Four

This beautiful executive-function rule was first introduced by Debbie Hughes-Hallet [9]. It says that functions should be explored using four modalities. Every function should be understood:

- Symbolically, for example, an algebraic equation,
- Computationally, for example a function table,
- Graphically, whether manually or by calculator, and
- Verbally.

Problem Type	Typical Verbal Problem
Projectile / Gravity	Zacchini performed the human cannonball stunt for Ringling brothers. The tip of the canon rose 15 feet off the ground and the total horizontal distance traveled was 175 feet. When the canon is aimed at a 45-degree angle, its equation is parabolic. Determine the quadratic equation corresponding to the data given; find the maximum height attained [17].
Suspension bridges	A suspension bridge with weight uniformly distributed along its length has twin towers that extend 100 meters above the road surface and are 400 meters apart. The cable (that suspends the bridge) touches the surface at the center of the bridge. Find the height of the cables at a point 100 meters from the center [26].
Profit	A company charges $200 for each box of tools on orders of 150 or less. The cost to the buyer is reduced by $1 for each box ordered in excess of 150. What is the maximum revenue and what size order achieves this maximum [26]?
Geometry: Pythagorean triangles, squares, circles, rectangles and fences	Find the dimensions of a fence of maximum area, if 600 feet of chain link are used to enclose a rectangular region and to further subdivide the region into two, equal, smaller regions by placing a divider fence in the middle of the rectangle parallel to two of its sides [2].
Time rates/work	To travel 60 miles, it takes Sue riding a moped 2 hours less than it takes Ann riding a bicycle to travel 50 miles. Sue travel 10 miles per hour faster than Ann. Find the rate of each girl's ride [10].
Number Theory	The sum of two numbers is 5; their product is 6. Find the numbers.
Parabolic reflectors	If a parabolic reflector is 20 cm in diameter and 5 cm deep, find its focus, the point where incoming parallel rays reflect to [25].

Table 1: Seven major classes of verbal problems that can be modeled and solved using quadratic equations.

To illustrate the rule of four, consider the task of solving an equation. Equations are traditionally solved symbolically by algebraic manipulations. But they can also be solved graphically and by inspection of function tables.

2.3. Quadratic Parameters

The analysis of a domain based on multiple parameters is an application of executive function. The quadratic parameters are *vertex, root, extremum, number of roots, axis of symmetry, concavity,* and *standard polynomial forms.* To say that these are parameters simply means that responses to questions about quadratic functions depend on the value of these parameters. For example:

- A quadratic extremum is a maximum or minimum depending on whether the quadratic function is concave up or down.
- Table 2 lists four quadratic polynomial standard forms. Each standard form is useful for answering a specific question as shown in the table.
- The number of real roots of a quadratic equation can be ascertained by examining the sign of the discriminant b^2-4ac.
- There are a variety of data indications that can be used to create a quadratic function. For example, i) vertex and point, ii) roots and an extremum.

Form Name	Symbolic form	What is form best for?
Vertex form	$a(x-h)^2+k$	What are the x and y values for maximum/minimum?
Root form / Factored form	$a\,(x-r)\,(x-s)$	What are the roots/ zeroes?
Standard form	$ax^2 + bx + c$	Is the vertex of the quadratic function a maximum or minimum?
Rate Form	$a/(x-t) + b/(x-s) = c$	Work rate problems—that is, problems asking for the rate of combined activities each with its own rate—are naturally modeled using this form.

Table 2: Four quadratic polynomial forms.

2.4. Projects

Almost anything in this section can be transformed into a project. A project addresses executive function by combining the mental and physical. Examples of projects are the following: i) measuring the time for dropped balls to reach the ground [17]; ii) measuring the rate at which two pipes together empty a tank; iii) finding the graph of suspension from two parallel towers

2.5. Goal- Setting

Recall that goal-setting refers to the breaking of a complex task, such as solving a quadratic equation, into a sequence of simple tasks each of which is challenging, achievable in a short time, and clearly defined [14,15]. Numerous studies, particularly in industry, show that when employees are given clear, challenging tasks, achievable in a short time, their productivity increases [15]. Table 3 applies goal-setting to the task of solving a quadratic equation.

To clarify the use of Table 3, consider a student who wishes to learn to solve any quadratic equation:

(i) The student would first be given several problems of the form $x^2=a$, the first row in the table. In a short time, a student would quickly learn that when appropriate, two roots must be given.

(ii) Next, the student would be given equations of the form $ax^2-b=0$, the second row in the table. The student would be instructed, possibly with illustrative examples, to solve these problems using linear methods (by adding and multiplying the same quantity to both sides of the equation). This subtask could be learned quickly, that is, it is achievable in a short time with the student being able to achieve complete mastery. Note especially that this step might also include equations of the form $ax^2+b = c(dx^2+e)$, equations which require applying linear methods.

(iii) The teacher would then continue presenting equations of the forms in the last two rows of Table 3 until all quadratic equations are mastered.

The point of goal-setting, is that each step can be mastered in a short time, has clearly defined terminal states, but is challenging. While minimal class time must be spent on each step, it would enable each student to be able to solve any quadratic equation.

Equation to focus on	What is the challenge and novelty?
$x^2=a$	Students are used to solving linear equations where there is one solution. With quadratic equations, students must learn that solutions typically come in pairs.
$ax^2=b$	Students must *transfer* the methods by which linear equations are solved—adding the same number to both sides or multiplying both sides by the same number— to quadratic equations, and combine this skill with taking square roots.
$x^2+2b = c$	Students must learn a new skill, completing the square.
$ax^2+bx+c=0$	Students must combine the three skills previously learned: i) applying linear methods (for example dividing by a); ii) completing the square, and iii) identifying root pairs if applicable.

Table 3: Goal-setting strategy for solving the quadratic equation

2.6. Omissions and Miscellaneous Topics

Several topics—e.g. trinomial factoring, the quadratic formula, and complex numbers—are conspicuously absent from the discussion in this section. The reason for these omissions will be dealt with in Section 5.

2.7. Attribution Theory and Self-Efficacy

This section has covered application of the first two pillars of good pedagogy, executive function and goal-setting, to the quadratic equation. Self-efficacy is primarily achieved through past performance successes [3]. Attribution theory advocates effort, for example by practicing exercises, as a means of mastery. Thus, the pillars of attribution theory and self-efficacy are adequately met if textbooks provide adequate exercise resources and illustrative examples for each concept taught.

3. TEXTBOOK EVALUATION

Eight textbooks [2,6,10,12,17,26,27,28] were selected for review. Five of them [2,6,10,26,27] were written in the early 1990s or late 1980s. The remaining texts [12,17,28] were published between 2001 and 2009. Recall from Section 1 that the curriculum standards were published in 1989 [20] while the Process Standards were published in 2000 [23] and the SMP were published in 2010 [24]. In the sequel, [2,6,10,26,27] will be referred to as pre-2000 texts and [12,17,28] as post-2000.

These eight texts were selected because they are good texts, typical texts, and have varied approaches. Each text, was at some point selected for classroom use, and thus was selected over other competing choices. The eight texts have a variety of:

- *Target audiences* (high-school [12,17,28] and college [2,6,26,27]. Some college texts were field-tested by high schools (e.g. [10]).
- *Focuses* (the section or chapter titles indicate this: quadratic functions, quadratic equations, problem solving [12,28], and an investigative approach [17] indicating an emphasis on projects; two of the textbooks [12,28] have separate sections for the quadratic function and equation).
- *Periods* (as indicated there are five pre-2000 textbooks and three post-2000 textbooks written between 2001-2009).

Although some of these texts are used as precalculus texts at the college level, the topic quadratic equation/function is a high-school topic and therefore it is meaningful to apply the Process Standards and the SMP to them.

The immediate investigative question in this section is to ascertain *to what extent do these textbooks focus on the quadratic verbal problem classes listed in Table 1.* Table 4 presents a scoring of each book. An empty square means no exercises were given in the text illustrating that quadratic problem class (even if the problem class is mentioned). Asterisks indicate the presence of exercises in the problem class. Table 4 summarizes both the total number of verbal problems and the percentage of problem classes from Table 1 covered by each text.

Problem/ Text	2	6	10	12	17	26	27	28
Projectile	**	**		**	**	**		**
Suspension						**	**	
Profit	**	**	**	**	**	**	**	**
Geometry	**		**	**	**	**	**	
Time rates			**			**		**
Number Theory						**		
Reflectors								
Percentage of Classes covered	43	29	43	43	43	86	43	43
Number verbal problems	8	10	23	18	15	22	16	26

Table 4: Presence of problem classes (Table 1) in various books. Brackets are omitted in the first row; the numbers refer to references (e.g. [2], [6]). Similarly, percentage symbols are omitted in the next to last row (e.g. 43%, 29%).

3.1. Discussion of Results

Many of the textbooks only give a dozen verbal problems and only cover about half the problem classes. [26], a pre-2000 text, covers 86% of the quadratic problem classes in Table 1 and presents about 2 dozen verbal problems. Contrastively, [12,17,28], post-2000 texts, only cover 43% of the problem classes.

What is the significance of these findings? As pointed out in Section 1, verbal problems:

- Strongly involve executive function, one of the four pillars of good pedagogy. Recall, executive function refers to using multiple modalities of the mind and/or multiple parameters. Thus, using the verbal and symbolic area of the mind is a strong manifestation of executive function.
- Meet most of the Process Standards and most of the SMP.

This paper is basically making the following two arguments:

- If an instructor is interested in good pedagogy as defined by Hendel [8], then the pillar of executive function should be the primary focus. Consequently, a course is superior if all seven quadratic problem classes of Table 1 are presented and practiced until mastery is achieved. Such a course design also exposes students to mathematics as a discipline that continuously interacts with the real world.
- Similarly, if the instructor's goal is to meet pedagogic excellence as defined by the Process Standards [23] or the CCSSM [24], then the skillful use of a variety of verbal problems, whose treatment is rich in these standards, should be a primary focus.

The main conclusion and recommendation of this paper is that *instruction should primarily focus on verbal problems and all seven problem classes should be included in any course design.*

3.2 Other Aspects of Quadratic Functions

Section 2 listed five broad areas required to teach the quadratic function. Most of the eight textbooks reviewed dealt with most of these areas adequately. Hence, these other areas are not used to differentiate between the texts. Some pedagogically interesting aspects of these books are as follows:

i) [17] particularly emphasizes an investigative approach. As already indicated, projects address executive function by integrating the mental and physical. They also frequently deal with modeling phenomena covered by verbal problems. However, [17] only has projects for a few of the areas mentioned in Table 1, notably falling objects. The recommendation of this paper can be extended to textbooks that focus on projects provided all seven areas mentioned in Table 1 are covered by a project.

ii) The idea of solving equations by graphs and function tables is lacking in the pre-2000 books but present in the post-2000 books. This is undoubtedly due to the influence of Hughes-Hallet [9]. However, even the pre-2000 books mention the duality of graphs and functions.

iii) Although all the books adequately mention the quadratic parameters, [28] beautifully formulates these ideas in the form of theorems thus giving them extra emphasis (e.g. the *graph translation* theorem; the *absolute value-square root* theorem, the *binomial square* theorem, etc.)

4. GOAL SETTING

The ten criteria for proper goal-setting [15] are not explicitly part of the Process Standards or the SMP. Yet study after study (mostly in industry settings, but also in teaching settings) shows that when complex tasks are broken up into clear, challenging subtasks, achievable in a short time, with opportunity for feedback, productivity increases. Thus, this paper recommends *revision of the standards to explicitly include the ten criteria of proper goal-setting in the Process Standards and the SMP.*

Although all the textbooks mention the subtasks listed in Table 3, none of the textbooks mention them in an *organized sequential* manner. None of the texts give exercises focused specifically on each subtask, though the books are quick to give exercises which mingle the subtasks. Students, particularly, the weaker students, need the opportunity to have a bird's-eye view of all skills needed for solving the quadratic equation, and the available resources to master each skill separately.

4.1. Written Solutions

Process goal c) as well as SMP iii) require students to be able to communicate, argue and critique. Students solving an equation, should be able to list the steps involved and relate each one to some basic concept as done in Table 3. This can be accomplished through writing exercises. Figure 1 presents a sample written response to solving a quadratic equation.

> *I solve the quadratic equation, $4x^2 - 16x + 15 = 0$.*
>
> *I know from linear methods, that both sides of the equation can be divided by the same quantity. This will enable me to transform the given equation to one with leading coefficient 1, which is easier to solve. I simply divide both sides by 4 to obtain, $x^2 - 4x + 3.75 = 0$. Furthermore, linear theory teaches us to separate variables and constants. I can achieve this by subtracting 3.75 from both sides of the equation. I obtain $x^2 - 4x = -3.75$.*
>
> *Solving quadratic equations requires use of non-linear techniques. In this case, I must complete the square on the left side. The binomial factor $(x-2)$ when squared yields $x^2 - 4x+4$. I can complete the square by adding 4 to both sides of the equation. I obtain $(x-2)^2=x^2 - 4x +4= -3.75+4=0.25$.*
>
> *Solving the equation $(x-2)^2=0.25$, can be accomplished by taking square roots of both sides. Care must be taken to allow for positive and negative square roots. I obtain the two solutions $x-2 = 0.5$ and $x-2=-0.5$*
>
> *Linear methods now give the final answer: $x = 2.5$; $x=1.5$*

Figure 1: Model written answer of solving a quadratic equation. The written answer meets the Process Standard of communication and the SMP standard of argumentation. Written exercises like this should be a component of any course.

Figure 1 is *one* method of written communication and is perhaps overly leisurely. An alternate method is presented in the next section. A 3rd alternative is presented by [12,28] which accomplish written communication through a step-by-step algebraic derivation with annotated phrases explaining each step. There is no unique correct written solution; rather, the emphasis is on each student writing some multi-step solution.

4.2. An Alternate Goal-Setting

[26,28] gives an alternative goal-setting approach worth mentioning. To clarify this approach, note that Table 3 presents goal-setting for *computing zeroes* of a quadratic equation. *Computing zeroes* is one approach to graphing.

An alternative approach, presented by [26], is to graph based on the *axis of symmetry*. If a quadratic function is given by the equation $a(x-h)^2+k$, then there is an axis of vertical symmetry on the line $x = h$, and the vertex of the parabola is at (h, k). The parabola has a minimum (maximum) at $x=h$ if $a > 0$ ($a < 0$).

Thus, a suitable alternative to the *zeroes* approach of graphing a parabola is the *vertex-extremum* approach. This approach is equally valid to the *zeroes* approach. Indeed, if one is interested in the extremum, then the vertex form is superior to the zero form. The approach of [26,28] to graphing is presented in Table 5 with proper goal-setting.

[26,28] list all four of these stages. Furthermore, [26] provides 13 exercises where graphs are drawn using *contraction, dilation, horizontal* and *vertical translation.* [28] explicitly coins a new term and presents *the translation graphing* theorem to facilitate this approach.

5. OMITTED TOPICS

Section 2 omitted factoring and the quadratic formula, two topics commonly taught in connection in the quadratic module. This paper argues that these topics should only be covered lightly in a course. In fact, this paper argues for dropping them.

Equation to graph	What is the challenge and novelty
$y=x^2$	Students are used to graphing linear equations. This is their first exposure to a non-linear function. Key characteristics of the parabola are its extremum and its axis of symmetry.
$y=ax^2$	An important concept, in fact mentioned by several other books besides [26,28], is that whether a parabola has a maximum or minimum depends on the sign of a. Another important point is that a has the effect of either a *contraction* or *dilation* depending on whether a is bigger or less than 1.
$y=a(x-h)^2$	The graph of $y=a(x-h)^2$ can be obtained from the graph of $y=ax^2$ by a *translation* of h units along the x axis.
$y=a(x-h)^2+k$	The graph of $y=a(x-h)^2+k$ can be obtained from the graph of $y=a(x-h)^2$ by a *vertical translation* of k.

Table 5. Goal-setting for the graphing of a quadratic equation.

To defend dropping the quadratic formula, recall that in Figure 1 we show a model written answer to solving a quadratic equation using a written exercise. In fact, Figure 1 goes through the steps needed to derive the quadratic formula. This model answer addresses both executive function by employing several different techniques as well as goal-setting by developing the solution over several substeps. Contrastively, the quadratic formula approach is just an instant plug-in with minimal involvement of executive function.

While there may indeed be practical reasons for including the quadratic formula in a course, the emphasis in this paper is that by itself, the quadratic formula does not meet the requirements of pedagogic excellence such as executive function or goal-setting. Indeed, if one's goals are the four pillars, or if one's goals are the Process Standards or the SMP, these goals can be achieved without factoring and the quadratic formula; in fact, factoring and the quadratic formula might hinder learning. Thus, the inclusion of the quadratic formula by itself does not enhance the pedagogic excellence of the course. If it is still needed in the course, it is needed for *other* reasons. In summary, this paper recommends omission or diminished inclusion of the quadratic formula with the main teaching focus on pedagogic excellence. Section 6 shows how to include diminished use of the quadratic formula in a course.

Similarly, this paper argues against including factoring in the quadratic curriculum. It can be replaced by knowledge and usage of the *factor* form listed in Table 2, $a\ (x\text{-}r)\ (x\text{-}s)$, which can be obtained *after* one solves for the zeroes.

The driving force for advocating removing factoring is its complexity. There is no elegant goal-setting for factoring (unless one makes it an entire module by itself). To see this, consider a simple example such as $x^2\text{-}5x\text{+}6\text{=}0$. This is a rather simple case of factoring since the coefficient of x^2 is 1. But even in this case the factoring is complex. One must find two numbers whose sum is 5 and whose product is 6. The only way to do this is to list alternatives: (1,6), (2,3). For each alternative, one must check if their sum is 5 and their product 6. This is complex. Thus, only the stronger students fully succeed in factoring while the weaker students make errors. Inclusion of a complex topic in a module that differentiates stronger and weaker students indicates a lack of proper goal setting. The differentiation can damage weaker student self-efficacy and this weakened self-efficacy could ripple through the rest of their learning.

As with the quadratic formula, the emphasis of this paper is not on total removal but rather on perceiving factoring as not enhancing pedagogic excellence.

6. PRACTICAL CONSIDERATIONS

In this section, we present an approach, the *mixture approach*, for both achieving pedagogic excellence as well as achieving the types of drill mastery traditionally needed in some courses by diminished inclusion of such topics as the quadratic formula and factoring.

First, we indicate the motivation for the *mixture approach.* This paper has focused on pedagogical excellence. But an equally important aspect of education is operational, the identification of what a student completing the course can do. This is particularly true in mathematics which frequently hosts service courses for other disciplines.

For example, it is not sufficient for students completing a mathematics prerequisite to a statistics course to be able to write essays and be able to derive results. A student completing a mathematics prerequisite should be able to do certain computations quickly since these computations might be part of a larger process. In such a case, it is the larger process that should be written and explained; the individual components of the larger process might be taken as is without a need for explanation or derivation.

Here is a further simple illustration using the quadratic formula. Suppose a course, say an introductory algebra course, is a prerequisite for a ballistics course which studies missile trajectories and the interaction between a starting and ending location. The missile trajectory course is possibly interested in complex missile problems involving destroying, with precision, enemy centers using missiles fired from a significant distance. The quadratic function theory is one component of the solution to such problems. What is needed to solve these complex problems is on-the-spot instant solution of quadratics. Exposition would unnecessarily lengthen solution of the missile problems. What is needed is operational quickness of solution.

How then can we reconcile these two opposing educational requirements, the requirement of understanding as evidenced by an assessment of expository writing vs. the requirement of operationality as evidenced by the ability to quickly and accurately perform certain computations?

This paper suggests a *mixture approach* in both teaching and assessment. The *mixture approach* would devote a certain percentage of the class to methods of pedagogic excellence; the remaining percentage of the class would be devoted to operational considerations, the ability to solve problems quickly and accurately. This would also apply to assessment. A typical test would have a performance component which counts a certain percentage while the remainder of the test would assess understanding of basic principles.

The percentage allocations would depend upon importance, needs, and time resources. Both extremes, say 90% performance vs. 10% performance, might be useful depending on the sequence of courses and time available. Such an approach would allow addressing both pedagogic and operational needs using a flexibly adjusted percentage.

7. CONCLUSION

This paper has advocated a specific method of teaching the quadratic equation that is consistent with the four pillars of good pedagogy and also consistent with the NCTM Process Standards and the CCSSM SMP. This paper recommends:

- Primary emphasis should be on verbal problem solving.
- Verbal problem solving can be enhanced if the richest diverse set of verbal problems is presented, each of which has its own quadratic form and its own methods.
- Project activities are acceptable as a substitute for verbal problems provided a diversity of project activities comparable to the variety of verbal problems is used.
- Formula memorization, such as the quadratic formula, should be replaced or supplemented with mini-essay writing, emphasizing derivation, argumentation and communication. If the course has a service component, then a *mixture approach* should be used addressing both needs of memorization and quick performance as well as expository writing and understanding.
- Excessively complex skills such as factoring should be left out of the course or lightly approached as a consequence of finding zeroes. If the course has a service component, then a *mixture approach* should be used addressing both needs of memorization and quick performance as well as expository writing and understanding. In such a case, the *finding zeroes* approach to factoring is useful for purposes of understanding while a more traditional *practice and drill approach* would be used to create factoring skill.
- The NCTM process standards and the CCSSM SMPs should be augmented to explicitly include the 10 requirements of good goal-setting.

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