Cumulative Impact of Inhomogeneous Channels on Risk

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ABSTRACT

This paper considers flows of containerized traffic in a cascade of channels with diverse risk characteristics. Each channel is characterized by a probability distribution function relating the probability of loss being less than a given value to the magnitude of the loss. The cumulative impact of cascading channels is then evaluated as a closed form solution in terms of the characteristics of the constituent channels with dissimilar risk characteristics. The results presented in this analysis can be used to shape the risk characteristics of individual channels through, for example, additional investment in order to maximize the impact of such investments.

Keywords: containerized traffic, risk characteristics.

1. INTRODUCTION

As businesses become increasingly global in scope, the flow of cargo, specifically containerized cargo, becomes a critical need. The safe and efficient movement of such cargo is becoming a national priority because of the impact of safe and uninterrupted flow of such cargo on national security as well as on economic growth of the nation [1, 2]. Containerized traffic travels over a variety of transport channels, including roads, the open sea, as well as air. The flow of containerized traffic will, generally speaking, also encounter a variety of gates that include national boundaries, customs and other government-mandated check points. Each of the modalities of transportation and the gates encountered by the container during its transit from the source to the destination presents a varying risk profile [3]. Each would impact the end-to-end risk characteristics of containerized traffic flow in complex ways.

This paper develops a closed form solution relating the end-to-end risk behavior of containerized traffic flow in terms of the characteristics of each of the channels or gates. Understanding the end-to-end risk characteristics is important from the perspective of business because the business needs to develop a predictable model for risk in order to sufficiently insure its cargo and factor the costs of such insurance in the pricing model. From a national security perspective, each government or national security agency needs to carefully weigh in the costs of improving safety and security against the predicted enhancement in attaining such security on an end-to-end basis. While the impact of an investment toward the improvement in the characteristics of a single channel or gate might be easily understood, this understanding is not sufficient in terms of evaluating the impact of that investment from an overall risk mitigation perspective when several channels in sequence are involved. Accordingly, it might not yield the best 'bang for the buck' invested. This paper characterizes the end-to-end risk profile in terms of the characteristics of each of the constituent elements in order to maximize the impact of investment toward improving the end-to-end risk characteristics.

The conventional way of understanding risk is in terms of the expectation of loss which is a product of the probability of loss and the average amount of loss. This information is insufficient if one were to assess the probability of loss remaining bounded within a predefined threshold. This is what we address in this paper, especially under the scenario that a cargo goes through when it traverses a number of channels that are inhomogeneous in their risk characteristics [4]. We first address the risk characteristics of a single channel followed by two channels. We then generalize the results to a number of cascaded channels with inhomogeneous risk characteristics. The findings of our investigation are illustrated through a number of examples.

2. THE SINGLE CHANNEL MODEL

Our model defines the risk characteristics of each intermediate traffic channel and each gate encountered by the in-transit cargo individually. We assume that the risk characteristic of each element (whether a channel or a gate) is independent. The risk characteristics of each element are defined by a single variable λ which is exponentially distributed [5].

The probability density function of this variable is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, x \ge 0 \\ 0, x < 0 \end{cases} \tag{1}$$

The cumulative distribution function is given by:

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, x \ge 0 \\ 0, x < 0 \end{cases} \tag{2}$$

An interpretation of Eq. (2) would be that the probability of a loss of magnitude x or less is F(x). The boundary conditions of Eq. (2) are verified by the fact that all losses are bounded by

infinity and zero. Using the properties of the exponential distribution, the mean loss of a channel $=\frac{1}{\lambda}$. Figure 1 shows the distribution function for three different values of λ =0.5, 1, and 1.5. The curve with a higher value of the mean loss (lambda=0.5) rises more slowly as expected.
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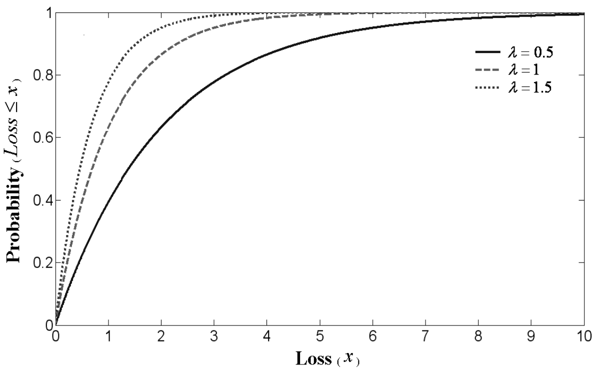

Figure 1: Loss Characteristics of a single channel

3. THE TWO-CHANNEL MODEL

We now consider a two-channel model, say, consisting of road and air transport channels [6]. The probability density function of the two channels is defined as:

$$f_1(x) = \lambda_1 e^{-\lambda_1 x} \tag{3}$$

$$f_2(x) = \lambda_2 e^{-\lambda_2 x} \tag{4}$$

The combined probability density function f(x) of both the channels can be evaluated by convolving the two constituent probability density functions $f_1(x)$ and $f_2(x)$ [7, 8]. We have,

$$f(x) = f_1(x) \otimes f_2(x) = \int_{-\infty}^{\infty} f_1(\xi) f_2(x-\xi) d\xi \tag{5}$$

After some simplification, (See Appendix A), Eq. (5) can be expressed as,

$$f(x) = \begin{cases} \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 x} - e^{-\lambda_2 x}), (\lambda_1 \neq \lambda_2) \\ \lambda^2 x e^{-\lambda x}, (\lambda_1 = \lambda_2 = \lambda) \end{cases} \tag{6}$$

$$F(x) = \begin{cases} 1 - \frac{1}{\lambda_2 - \lambda_1}(\lambda_2 e^{-\lambda_1 x} - \lambda_1 e^{-\lambda_2 x}), (\lambda_1 \neq \lambda_2) \\ 1 - (1 + \lambda x) e^{-\lambda x}, (\lambda_1 = \lambda_2 = \lambda) \end{cases} \tag{7}$$

From Eq. (7), we notice that λ_1 and λ_2 are interchangeable. Therefore, the sequence of the channels does not affect the end-to-end loss. In other words, the end-to-end loss characteristic is determined entirely by the loss characteristics of each channel and is independent of their sequence.

Examples:

In Figure 2, we present the loss characteristics of four different situations, each with two channels and the following loss characteristics.

Case 1: $\lambda_1 = 0.75, \lambda_2 = 1.25$
Case 2: $\lambda_1 = 0.5, \lambda_2 = 1.5$
Case 3: $\lambda_1 = 0.25, \lambda_2 = 1.75$
Case 4: $\lambda_1 = 1, \lambda_2 = 1$

Note that in each case the sum of λ_1 and λ_2 has been kept at a constant value, namely, $\lambda_1 + \lambda_2 = 2$.

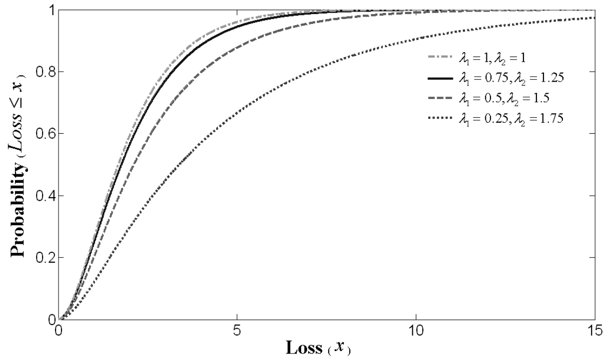

Figure 2: Loss Characteristics of two channels in tandem

It is evident from Figure 2 that closer values of λ_1 and λ_2 result in curves that represent better end-to-end loss characteristics, i.e., the curves rise faster than those where λ_1 and λ_2 widely vary.
We note in passing that the mean value of loss of two channels in tandem is equal to $\frac{1}{\lambda_1} + \frac{1}{\lambda_2}$. This sum will obviously increase as λ_1 and λ_2 diverge while their sum remains constant. (See Appendix B). In other words, the cumulative loss characteristics of two channels in tandem are consistent with the loss experienced by each of the two channels.

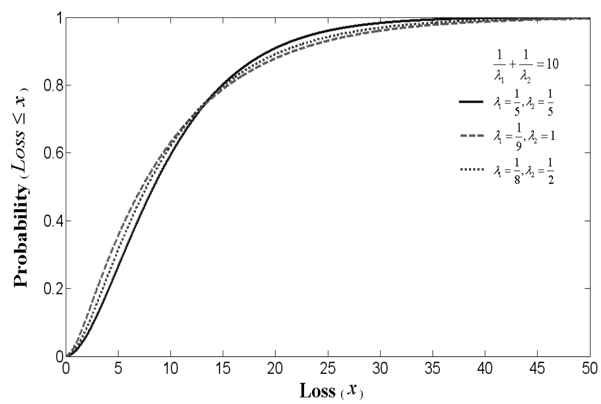

Figure 3: Loss Characteristics of two channels with the same end-to-end mean loss

Figure 3 plots a number of curves for two channels in random where we have kept the sum $\frac{1}{\lambda_1} + \frac{1}{\lambda_2}$ a constant while varying the individual λ_1 and λ_2. It can be observed from Figure 3 that, as long as the mean end-to-end loss experienced is kept to be a constant, the variance in their cumulative loss characteristics is moderate.

We can now ask ourselves the following question: Given identical end-to-end mean loss, is it preferable to have a single channel or two channels in tandem? A surprisingly elegant result is presented in Theorem 1.

Theorem 1: Compared to two identical channels in tandem, each having a loss parameter equal to λ, a single-channel with the parameter $\frac{\lambda}{2}$ has superior cumulative loss characteristics up to a mean loss value that can be numerically evaluated. Beyond this point, the two-channel model with identical mean loss is superior.

Proof:

We note that the mean loss value for each model is identical since $\frac{1}{\lambda}+\frac{1}{\lambda}=\frac{1}{\frac{\lambda}{2}}$. We also have, for its single-channel model,

$$F_1(x)=1-e^{-\frac{\lambda}{2}x} \tag{8}$$

And for the two-channel model,

$$F_2(x)=1-(1+\lambda x)e^{-\lambda x} \tag{9}$$

For very small values of x, we have

$$F_1(x)\approx\frac{\lambda}{2}x-\frac{\lambda^2x^2}{8} \tag{10}$$

$$F_2(x)\approx\frac{\lambda^2x^2}{2} \tag{11}$$

And therefore, for such values of x,

$$F_1(x)>F_2(x)$$

The two loss curves intersect when $F_1(x)=F_2(x)$, i.e., when from Eq. (8) and (9)

$1-e^{-\frac{\lambda}{2}x}=1-(1+\lambda x)e^{-\lambda x}$, or when

$$(1+\lambda x)e^{-\frac{\lambda}{2}x}-1=0 \tag{12}$$

Eq. (12) can be numerically evaluated.

For large values of x, we have,

$$F_2(x)-F_1(x)=e^{-\frac{\lambda}{2}x}-(1+\lambda x)e^{-\lambda x} \tag{13}$$

The ratio of the two terms on the R.H.S. of Eq. (13) can be written as,

$$\frac{e^{-\frac{\lambda}{2}x}}{(1+\lambda x)e^{-\lambda x}}=\frac{e^{\frac{\lambda}{2}x}}{1+\lambda x} \tag{14}$$

The problem then becomes to compare the relative values of $e^{\frac{\lambda}{2}x}$ and $1+\lambda x$. For a given value of λ, we further note that since the slope of an exponential function, $\frac{\lambda}{2}e^{\frac{\lambda}{2}x}$ increases with x, while the slope of a straight line is constant λ, there cannot be more than two points of intersection between a straight line and an exponential curve. The two channel model thus has better loss characteristics than the single channel model at higher values of loss.

4. THE N-CHANNEL MODEL

For n-channel model [9], we also have two cases. One supposes that each channel has the same λ, then

$$f_n(x)=\frac{x^{n-1}}{(n-1)!}\lambda^n e^{-\lambda x} \tag{15}$$

$$F_n(x)=1-\sum_{i=0}^{n-1}\frac{(\lambda x)^i}{i!}e^{-\lambda x} \tag{16}$$

The other supposes λ, as

$$f_3(x)=\frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 x}+\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}e^{-\lambda_2 x}$$
$$+\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 x} \tag{17}$$

$$f_n(x)=\lambda_1\lambda_2\sum_{i=1}^{n}\frac{1}{\prod_{j=1,j\neq i}^{n}(\lambda_j-\lambda_i)}e^{-\lambda_i x} \tag{18}$$

$$F_n(x)=1-\lambda_1\lambda_2\sum_{i=1}^{n}\frac{1}{\lambda_i\prod_{j=1,j\neq i}^{n}(\lambda_j-\lambda_i)}e^{-\lambda_i x} \tag{19}$$

5. CONCLUSION

This paper has provided a closed form solution to the general problem of assessing the probability of bounding loss in a cascade of channels when the risk characteristics of each channel, modeled as an exponential loss model, are known. The results obtained can be utilized to shape the loss characteristics of individual channel.

APPENDIX A

$$f(x)=f_1(x)\otimes f_2(x)=\int_{-\infty}^{\infty}f_1(\xi)f_2(x-\xi)d\xi$$
$$=\int_{-\infty}^{\infty}\lambda_1e^{-\lambda_1\xi}\lambda_2e^{-\lambda_2(x-\xi)}d\xi=\lambda_1\lambda_2e^{-\lambda_2x}\int_0^x e^{(\lambda_2-\lambda_1)\xi}d\xi$$
$$=\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_2x}(e^{(\lambda_2-\lambda_1)x}-1)=\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}(e^{-\lambda_1x}-e^{-\lambda_2x}) \tag{20}$$

For $\lambda_1=\lambda_2=\lambda$,

$$f(x)=\int_{-\infty}^{\infty}\lambda e^{-\lambda\xi}\lambda e^{-\lambda(x-\xi)}d\xi=\int_0^x\lambda^2e^{-\lambda x}d\xi=\lambda^2xe^{-\lambda x} \tag{21}$$

APPENDIX B

Let $\lambda_1+\lambda_2=C$ (a constant) (22)

We intend to show that $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$ is minimized when $\lambda_1=\lambda_2$.

Let $\lambda_2-\lambda_1=\Delta$ (23)

We have assumed, without loss of generality that $\lambda_2>\lambda_1$, i.e. $\Delta>0$. From Eq. (22) and (23), we have $\lambda_2=\frac{C+\Delta}{2}$ and $\lambda_1=\frac{C-\Delta}{2}$.

We now can write after some algebraic simplification,

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=\frac{4C}{C^2-\Delta^2} \tag{24}$$

For C and $\Delta>0$, it can be easily shown that (24) is minimized when $\lambda_1=\lambda_2$.

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