Generalized Tellegen Principle and Physical Correctness of System Representations

Václav Černý
Department of Cybernetics, University of West Bohemia
Plzeň, 306 14, Czech Republic
and
Daniel Mayer
Department of Theory of Electrical Engineering, University of West Bohemia
Plzeň, 306 14, Czech Republic
and
Josef Hrušák
Department of Applied Electronics, University of West Bohemia
Plzeň, 306 14, Czech Republic

***Abstract*— The paper deals with a new problem of physical correctness detection in the area of strictly causal system representations. The proposed approach to the problem solution is based on generalization of Tellegen's theorem well known from electrical engineering. Consequently, mathematically as well as physically correct results are obtained. Some known and often used system representation structures are discussed from the developed point of view as an addition.**

***Index Terms*— Power, Energy, Minimality, Equivalence**

I. Introduction

It is familiar that there are two basic approaches to system modelling. The first one consists in using mathematical formulas and physical tools (a causality principle, different forms of conservation laws, power balance relations, etc.) in order to describe appropriate system behavior. It has successfully been used in many fields of science and engineering so far. However, there are also situations where physical laws are not known or cannot be expressed in a proper mathematically exact form. In that case the second basic approach to system modelling can be turned. It is based on identification methods working in terms of experimentally gained data [1], [2]. It is possible to divide the methods into two groups: parametric and non-parametric, respectively. If any prior information about a system structure is not assumed then non-parametric methods are used for system identification. On the other hand, imagine that a physical structure of an investigated system is known then parametric methods will be used and subsequently more adequate results should be expected [3]. Unfortunately, any reliable explicit knowledge about a physical system structure is more likely an exception than a rule. Therefore, a system structure is mostly chosen on behalf of heuristic arguments and then it is verified whether obtained quantitative results are not in conflict with obvious qualitative expectations concerning regular system behavior and/or results of additional experiments performed on a real system.

The main aim of the contribution is to formulate a fundamental problem of physical correctness detection of system representations and in the sequel propose its possible solution. The approach starts from the assumption that any physically correct system representation should not be at variance with not only measured data but also a form of an energy conservation principle. It is shown in the paper that introducing the principle as the attribute of a causal system representation seems to be the most natural way as it can be done.

II. Tellegen's theorem and its generalization

In order to explain essential features of the theorem [4], consider an arbitrarily connected electrical network with n components and choose associated reference directions for branch voltages v_k and currents i_k. Let Kirchhoff's laws be given by the following equations:

$$Ai(t) = 0,\ Bv(t) = 0 \tag{1}$$

where A is a node incidence matrix, B is a loop incidence matrix and $i(t)$, $v(t)$ are defined as follows:

$$i(t) = [i_1(t), \ldots, i_n(t)]^T, v(t) = [v_1(t), \ldots, v_n(t)]^T. \tag{2}$$

Let the vectors $i(t)$, $v(t)$ be the elements of an Euclidean space E_n and invoke the inner product:

$$\langle i(t), v(t)\rangle = \sum_{k=1}^{n} i_k(t) v_k(t). \tag{3}$$

Let I be the set of all the vectors $i(t)$ and V the set of all the vectors $v(t)$ satisfying the equations (1).
Theorem 1: (Tellegen's theorem)
If $i(t) \in I$ and $v(t) \in V$ then it holds that:

$$\langle i(t), v(t)\rangle = 0. \tag{4}$$

Remark 1: It can also be expressed in the difference form:

$$\sum_{k=1}^{N}\sum_{j=1}^{N}(w_k v_{jk}^{'} - w_k^{'} v_{jk}) + \sum_{k=1}^{N}(w_k x_k^{'} - w_k^{'} x_k) = 0 \tag{5}$$

using for digital filter design where N is a number of nodes and w_k, v_{jk}, x_k, $w_k^{'}$, $v_{jk}^{'}$, $x_k^{'}$ are node variables, branch outputs and source node values of the first and second oriented graphs with the same topological structure [5], [6].

Remark 2: The sets I, V are orthogonal subspaces of E_n. Moreover, they $span\ E_n$.

It is worth noticing a close relation between physical correctness and Tellegen's theorem. It is also important to realize that the branch currents and voltages are chosen arbitrarily complied only with Kirchhoff's laws. It implies that different sets $\bar{I}$, $\bar{V}$ of the branch currents and voltages satisfying the laws can be selected and the relation:

$$\langle \bar{i}(t), \bar{v}(t) \rangle = 0,\ \bar{i}(t) \in \bar{I},\ \bar{v}(t) \in \bar{V} \tag{6}$$

still holds. The last deduction will be used later as motivation for introducing a group of system equivalence transformations on which generalization of Tellegen's theorem is based.

A. Generalized Tellegen principle

Consider the representation $R(S)$ of a system S in the form:

$$R(S) : \frac{dx(t)}{dt} = f[x(t), u(t)] \tag{7}$$

where $x(t) \in X$ is a state, $X \subset R^n$ is a smooth manifold and $f : X \rightarrow R^n$ is a smooth vector field parameterized by an input $u(t)$. Let $E : X \rightarrow R$ be a smooth scalar field. It is well known that the Lie derivative of the scalar field E with respect to the vector field f is defined as follows [7]:

$$\begin{aligned} L_f\{E[x(t)]\} &= \langle dE[x(t)], f[x(t), u(t)] \rangle = \\ &= \sum_{i=1}^{n} \frac{\partial E[x(t)]}{\partial x_i(t)} f_i[x(t), u(t)]. \end{aligned} \tag{8}$$

Remark 3: The only difference between the relations (4) and (8) is that both the factors are column vectors in the first one.

Theorem 2: (generalized Tellegen principle)

$$\begin{aligned} \exists E, f, E[x(t)] &= \sum_{i=1}^{n} E_i[x_i(t)], \\ \frac{dx(t)}{dt} &= f[x(t), u(t)] : L_f\{E[x(t)]\} = 0. \end{aligned} \tag{9}$$

Proof:

$$\forall E, f, dE \perp f : L_f\{E[x(t)]\} = 0. \tag{10}$$

Trivially,

$$E[x(t)] = E \Leftrightarrow \forall f : L_f\{E\} = 0. \quad \blacksquare \tag{11}$$

Corollary 1: (system representation structure)

$$\begin{aligned} \exists \varphi, T, T^{-1}, x^*(t) = T[x(t)], u(t) &= \varphi[v(t), x^*(t)] : \\ \langle x^{*^T}(t), \frac{dx^*(t)}{dt} \rangle &= 0. \end{aligned} \tag{12}$$

Proof: Consider a class of state and feedback equivalent representations:

$$\frac{dx^*(t)}{dt} = A^* x^*(t) + B^* K^* x^*(t) + B^* v(t) \tag{13}$$

$$y(t) = C^* x^*(t). \tag{14}$$

It can easily be shown that if the algebraic structure of the matrices A^*, B^*, C^*, K^* is as follows:

$$A^* = \begin{bmatrix} -\alpha_1 & \alpha_2 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\alpha_{k-1} & 0 & \alpha_k \\ 0 & \cdots & 0 & -\alpha_k & 0 \end{bmatrix}, B^* = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \beta_k \end{bmatrix}, \tag{15}$$

$$C^{*^T} = \begin{bmatrix} \gamma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, K^* = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \kappa_k \end{bmatrix},\ k = 2, \ldots, n \tag{16}$$

then the corollary 1 holds where $x^*(t) = Tx(t)$, $T = H_c^*[(A^*, B^*)] \cdot H_c^{-1}[(A, B)]$ in linear case. $\blacksquare$

III. Problem Formulation

Consider the external representation of a strictly causal system S in the form:

$$\frac{d^n y(t)}{dt^n} - u(t) = F[y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{n-1}y(t)}{dt^{n-1}}, \Theta] \tag{17}$$

where $\Theta = [\Theta_1, \ldots, \Theta_n]^T$ and a function F is known. The aim is to find an equivalence class of functions $\bar{S}$ for any given parametrization Θ in such a way that an internal representation induced by the equation:

$$\frac{d^n y(t)}{dt^n} + \bar{S}[\bar{x}(t), \bar{\Theta}] = u(t) \tag{18}$$

will not be in conflict with the signal power balance relation and a corresponding energy function will have the additivity property. Such an input-state-output representation will be called physically correct.

IV. Correct system representations

It seems to be quite obvious that two different issues should be distinguished. Mathematical correctness being equivalent to a state minimality property of a system representation and physical correctness closely related to energy additivity and a certain form of a signal power balance relation.

A. Mathematically correct system representations

Consider the representation $R(S)$ of a system S in the form:

$$R(S): \frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \quad (19)$$

$$y(t) = C(t)x(t) \quad (20)$$

where $x(t) \in R^n$ is a state, $u(t) \in R^r$ is an input, $y(t) \in R^p$ is an output, $1 \leq r \leq n$, $1 \leq p \leq n$, and matrices $A(t)$, $B(t)$, $C(t)$ are known. Assume that the system S is asymptotically stable and its representation $R(S)$ is of the minimum order n (controllable and observable). It implies that controllability and observability Grammian matrices W_c and W_o exist [8]. They are symmetric, positive definite and satisfy the following Lyapunov-like equations:

$$A(t)W_c(t)+W_c(t)A^T(t)+\frac{dW_c(t)}{dt}=-B(t)B^T(t) \quad (21)$$

$$A^T(t)W_o(t)+W_o(t)A(t)+\frac{dW_o(t)}{dt}=-C^T(t)C(t). \quad (22)$$

Those representations induce an equivalence class of minimum, controllable, observable and asymptotically stable representations given by the state equivalence conditions:

$$\bar{A}(t) = [T(t)A(t) + \frac{dT(t)}{dt}]T^{-1}(t) \quad (23)$$

$$\bar{B}(t) = T(t)B(t) \quad (24)$$

$$\bar{C}(t) = C(t)T^{-1}(t) \quad (25)$$

produced by the state transformations:

$$\bar{x}(t) = T(t)x(t),\ x(t) = T^{-1}(t)\bar{x}(t). \quad (26)$$

B. Physically correct system representations

Consider the time-invariant case of the representation (19), (20). A controllability Grammian matrix W_c at a time instant t is defined as follows:

$$W_c(t) = \int_{t_0}^{t} e^{A\tau}BB^Te^{A^T\tau}d\tau,\ 0 < t_0 < t \quad (27)$$

and has two properties:

$$W_c^T(t) = W_c(t) \geq 0 \quad (28)$$

$$Im[W_c(t)] = Im[H_c(A,B)] \quad (29)$$

where $H_c(A,B)$ is a controllability matrix. Supposing that $W_c(t)$ is invertible then the minimum energy input signal $u(t)$ exists and the minimum input signal energy E_u corresponding to state transfer from an initial state $x(t_0)$ to $x(t)$ is given by the relation:

$$E_u(t) = x^T(t)W_c^{-1}(t)x(t). \quad (30)$$

Consequently, the minimum input signal energy required for state transferring from the initial state $x(t_0)$ to $x(t_1)$ for $t \to \infty$ is given by the relation:

$$E_u = x^T(t_1)W_c^{-1}x(t_1) \quad (31)$$

under the assumption that the couple (A,B) is controllable and the asymptotical stability property is held. Then the Lyapunov equation:

$$AW_c + W_cA^T = -BB^T \quad (32)$$

expresses the form of an input-state energy transfer balance relation.

Similarly, an observability Grammian matrix W_o at a time instant t is defined as follows:

$$W_o(t) = \int_{t_0}^{t} e^{A^T\tau}C^TCe^{A\tau}d\tau,\ 0 < t_0 < t \quad (33)$$

and also has two properties:

$$W_o^T(t) = W_o(t) \geq 0 \quad (34)$$

$$Ker[W_o(t)] = Ker[H_o(A,C)] \quad (35)$$

where $H_o(A,C)$ is an observability matrix. Further, output signal energy E_y at a time instant t caused by an initial state $x(t_0)$ is given by the relation:

$$E_y(t) = x^T(t_0)W_o(t)x(t_0). \quad (36)$$

Subsequently, the largest observation energy produced by the initial state $x(t_0)$ for $t \to \infty$ is given by the relation:

$$E_y = x^T(t_0)W_ox(t_0) \quad (37)$$

supposing that the couple (A,C) is observable and the asymptotical stability property is held. Then the Lyapunov equation:

$$A^TW_o + W_oA = -C^TC \quad (38)$$

expresses the form of a state-output energy transfer balance relation. It follows from the energy additivity requirement:

$$E(t) = \sum_{i=1}^{n} \delta_iE_i[x_i(t)],\ \delta_i \neq 0 \quad (39)$$

that only those system representations can be accepted as physically correct whose Grammian matrices produced by the triplet (A,B,C) are diagonal and non-singular:

$$W_c \text{ or } W_o = W,\ W = diag\{\delta_1,\ldots,\delta_n\}. \quad (40)$$

V. Illustrative examples

Several simple examples are given in order to introduce natural concepts of physically correct and incorrect system representation structures.

Example 1: Let us have a real system with the following physical structure:

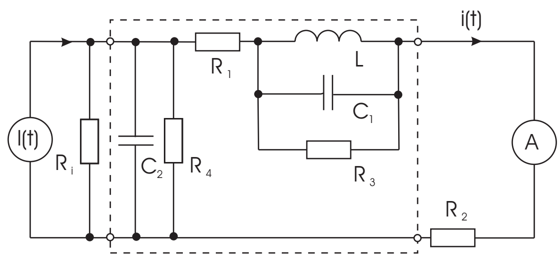

Fig. 1. Physical structure of a real system

where the physical meaning of the system parameters is known. On the other hand, numerical values of the parameters do not have to be known at all. Further, it is possible to get a mathematical model of the system as the physical state-space strictly causal representation:

$$R(S): \frac{dx(t)}{dt} = Ax(t) + Bu(t) \quad (41)$$

$$y(t) = Cx(t) \quad (42)$$

when proper state variables are injected and then energy conservation laws are used. Hence, the transfer function of the system is given as follows:

$$F(s) = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B = \frac{k(s)}{s^3 + a_1 s^2 + a_2 s + a_3} \quad (43)$$

where a_1, a_2, a_3 depend on the real system parameters. $k(s)$ is given by input signal characteristics.

Example 2: Let us suppose for now that the only information we have about the system is the transfer function without knowing any algebraic structure of the matrices A, B, C. A realization problem [9] is to find the matrices in such a way that the relation (43) holds. It is known that the solution of the problem is not unique because the specific algebraic structure of the matrices depends on the choice of the state variables. However, it is possible to determine the structure by carrying the transfer function $F(s)$ into the differential equation:

$$\frac{d^3 y(t)}{dt^3} + S(t) = ku(t) \quad (44)$$

where $u(t) = b_2 \frac{d^2 w(t)}{dt^2} + b_1 \frac{dw(t)}{dt} + b_0 w(t)$ and a function $S(t)$ defined as the scalar product of a parameter vector and a state vector:

$$S(t) = \langle \Theta, x(t) \rangle \Rightarrow S(t) = \sum_{k=1}^{3} \Theta_k x_k(t) \quad (45)$$

describes a relation between the state variables and structure of the system representation (41), (42). Since the parameter vector Θ is specified by $F(s)$, the natural choice of the state variables follows from the form of the function:

$$S(t) = a_1 x_1(t) + a_2 x_2(t) + a_3 x_3(t). \quad (46)$$

Thus, the matrices are the following and imply the topological structure of the system representation shown on the Fig. 2:

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, C^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (47)$$

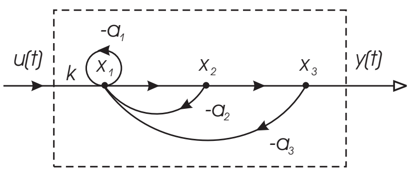

Fig. 2. Topological structure induced by A, B, C

Example 3: Let us take the orthonormal state transformation:

$$\bar{x}(t) = Tx(t), \; T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (48)$$

It produces another system representation with the different algebraic structure of the matrices:

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \bar{C}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (49)$$

but with the same topological structure shown on the Fig. 3:

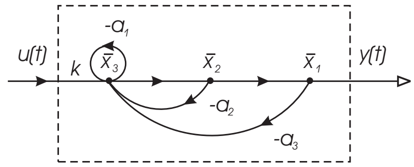

Fig. 3. Topological structure induced by $\bar{A}$, $\bar{B}$, $\bar{C}$

It is easy to verify that the resulting system representation structures are mathematically correct for any real values of the parameters. Nevertheless, the structures cannot be accepted as physically correct in the sense that both a signal power balance relation and signal energy additivity are required to hold simultaneously. To explain the situation, assume that the parameters values make the system be dissipative. The output signal power P_o is defined by the following relation:

$$P_o(t) = y^2(t) = \langle \bar{x}^T(t)\bar{C}^T, \bar{C}\bar{x}(t) \rangle = \bar{x}_1^2(t). \quad (50)$$

Then the output signal power balance relation reads [10]:

$$\frac{d\bar{E}_o[\bar{x}(t)]}{dt} = -P_o(t) \quad (51)$$

and holds if and only if the output signal energy $\bar{E}_o[\bar{x}(t)]$ has the form:

$$\bar{E}_o[\bar{x}(t)] = \bar{x}_1^2(t) + \frac{\Delta_1}{\Delta_2}[\Delta_1 \bar{x}_1(t) + \bar{x}_2(t)]^2 + $$

$$+ \frac{\Delta_1^2}{\Delta_3}[\frac{\Delta_2}{\Delta_1}\bar{x}_1(t) + \Delta_1 \bar{x}_2(t) + \bar{x}_3(t)]^2 \quad (52)$$

where $\Delta_1 = a_1$, $\Delta_2 = a_1 a_2 - a_3$, $\Delta_3 = a_3(a_1 a_2 - a_3)$ are Hurwitz minors. It is obvious that the energy function does not obey the additivity requirement.

Example 4: Consider the same system (43) but with another algebraic structure of the matrices. For now the state variables are chosen in such a way that the signal energy additivity requirement:

$$E_o^*[x^*(t)] = \delta_1 E_1^*[x_1^*(t)] + \delta_2 E_2^*[x_2^*(t)] + \delta_3 E_3^*[x_3^*(t)] \quad (53)$$

as well as the signal power balance relation hold. Let us take the following state transformation:

$$x_1^*(t) = \bar{x}_1(t) \quad (54)$$

$$x_2^*(t) = \Delta_1 \bar{x}_1(t) + \bar{x}_2(t) \quad (55)$$

$$x_3^*(t) = \frac{\Delta_2}{\Delta_1}\bar{x}_1(t) + \Delta_1 \bar{x}_2(t) + \bar{x}_3(t) \quad (56)$$

existing for $\Delta_1, \Delta_2, \Delta_3 \neq 0$. Then the energy function (52) becomes to:

$$E_o^*[x^*(t)] = x_1^{*^2}(t) + \frac{\Delta_1}{\Delta_2} x_2^{*^2}(t) + \frac{\Delta_1^2}{\Delta_3} x_3^{*^2}(t) \tag{57}$$

and the matrices of the system representation have the following algebraic structure:

$$A^* = \begin{bmatrix} -\alpha_1 & 1 & 0 \\ -\alpha_2 & 0 & 1 \\ 0 & -\alpha_3 & 0 \end{bmatrix}, B^* = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, C^{*^T} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{58}$$

where $\alpha_1 = \Delta_1$, $\alpha_2 = \frac{\Delta_2}{\Delta_1}$, $\alpha_3 = \frac{\Delta_3}{\Delta_1 \Delta_2}$. The corresponding topological structure of the representation is shown on the Fig. 4:

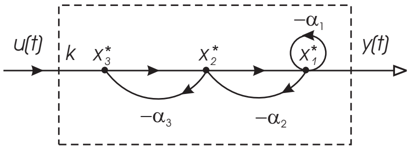

Fig. 4. Topological structure induced by A^*, B^*, C^*

Example 5: Let us have a real system with the following physical structure:

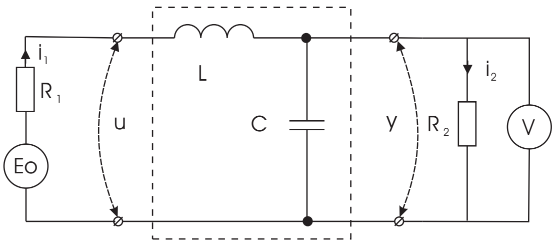

Fig. 5. Physical structure of a real system

Substituting the system parameters to (15), (16) we get for $n = 2$:

$$\begin{aligned} \alpha_1 &= \frac{1}{R_2 C},\ \alpha_2 = \frac{1}{\sqrt{LC}},\ \beta_2 = \frac{1}{\sqrt{LC}},\ \gamma_1 = 1, \\ \kappa_2 &= -R_1 \sqrt{\frac{C}{L}}. \end{aligned} \tag{59}$$

Then it holds that:

$$\begin{aligned} \langle x^{*^T}(t), \frac{dx^*(t)}{dt} \rangle &= -\alpha_1 x_1^{*^2}(t) + \beta_2 x_2^*(t) v(t) + \\ &+ \beta_2 \kappa_2 x_2^{*^2}(t) = P_D(t) + P_I(t) = 0 \end{aligned} \tag{60}$$

where input power P_I and output dissipation power P_D are given by the relations:

$$\begin{aligned} P_I(t) &= \beta_2 x_2^*(t) v(t) + \beta_2 \kappa_2 x_2^{*^2}(t) = \\ &= E_0 i_1(t) - R_1 i_1^2(t) \end{aligned} \tag{61}$$

$$P_D(t) = -\alpha_1 x_1^{*^2}(t) = -\frac{u_c^2(t)}{R_2} \tag{62}$$

$$x_1^*(t) = u_c(t)\sqrt{C}, x_2^*(t) = i_1(t)\sqrt{L}, v(t) = E_0\sqrt{C}. \tag{63}$$

Example 6: Physically correct structure of the system representation (13), (14) is shown on the Fig. 6:

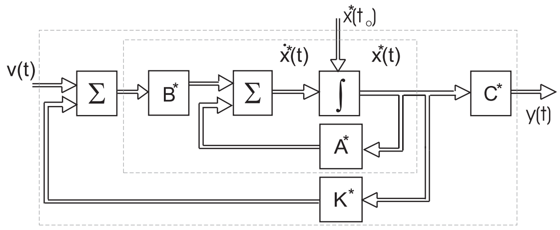

Fig. 6. Physically correct representation structure

VI. Acknowledgements

This work was supported by the Ministry of Education of the Czech Republic under the project No. 1M0567.

VII. Conclusions

The paper connects fundamental attributes of real-world situations (causality, physical correctness, different forms of conservation laws) with notions and results of electrical network theory (signal power, signal energy, Tellegen's theorem) as well as with basic approaches and concepts of general system theory (state minimality, equivalence relation, asymptotical stability, controllability, observability) and signal filtering [11], [12], [13].

References

[1] H. W. Sorenson, *Parameter estimation, principles and problems*. Marcel Dekker, New York, 1980.

[2] J. C. Willems, "Paradigms and puzzles in the theory of dynamical systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 3, pp. 259–294, 1991.

[3] D. Mayer and J. Hrušák, "On correctness and asymptotical stability in causal system theory," in *Proceedings of The 7th World Multiconference on Systemics, Cybernetics and Informatics*, vol. 13, Orlando, 2003, pp. 355–360.

[4] P. Penfield, S. Spence, and S. Dunker, *Tellegen's theorem and electrical networks*. MIT Press, Cambridge, 1970.

[5] A. Fettweis, "A general theorem for signal-flow networks with applications," *Archive Electronik Ubertragungstechnik*, vol. 25, pp. 557–561, 1971.

[6] R. E. Crochiere, "Digital network theory and its application to the analysis and design of digital filters," Ph.D. dissertation, Department of Electrical Engineering, MIT, 1974.

[7] J. P. Serre, *Lie algebras and Lie groups*. Springer-Verlag, Berlin, 2001.

[8] K. Zhou and J. C. Doyle, *Essentials of robust control*. Prentice Hall, New Jersey, 1998.

[9] R. E. Kalman, "Mathematical description of linear dynamical systems," *SIAM Journal on Control*, vol. 1, no. 2, pp. 152–192, 1963.

[10] J. Hrušák and V. Černý, "Non-linear and signal energy optimal asymptotic filter design," *Journal of Systemics, Cybernetics and Informatics*, vol. 1, no. 5, pp. 55–62, 2003.

[11] V. Černý and J. Hrušák, "On some new similarities between nonlinear observer and filter design," in *Preprints of The 6th IFAC Symposium on Nonlinear Control Systems*, vol. 2, Stuttgart, 2004, pp. 609–614.

[12] ——, "Non-linear observer design method based on dissipation normal form," *Kybernetika*, vol. 41, no. 1, pp. 59–74, 2005.

[13] ——, "Comparing frequency domain, optimal and asymptotic filtering: a tutorial," *Control and Intelligent Systems*, vol. 34, no. 2, pp. 136–142, 2006.