Meta-rules for Networks

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In memory of Eliano Pessa

ABSTRACT

This article proposes conceptual approaches for both the design and implementation of network linkages, that dependent on context, should respect interlinking topological and parametric constraints, possess formal properties, be compatible with configurational features and combine with an external meta-network as a service option. The purpose is to allow for alternative design and management of networks by adding flexibility, but to also enable design appropriate for highly specific applications.

Keywords: autocorrelation, linkage, meta, network, topologic.

1. INTRODUCTION

The aim of this article is to propose a framework as a guide to future and related research in the design and implementation of more flexible networks, that are also suitable for specific modeling and simulations. We mention features considered by the science of networks such as the cluster coefficient, local clustering coefficient, degree of a node, average degree, fitness, scale-freeness, small-world property, and properties considered by the percolation theory.

In this regard the additional concept of Meta-rules for networks is introduced, where meta is to regulate both topologically and parametrically, the network interconnective linkages established by the properties of these links and their possible weighting factors. As such, new versions are supposed to encompass meta-rules, in case context-sensitive, where their linkage is suitable to allow networks of the same design to operate in different ways, congenial to the particular application of interest. Furthermore, cases are considered where the meta-ruling occurs in contextual combination with another suitable meta-network of the same structure. In the latter case, a combination of meta-networks may be described by the same network but at different instances in time. Meta-rules may apply to static, evolving and dynamic networks, where the purpose is design flexibility. In introducing these meta-rules, modeling specific and complex phenomena as meta-ruled networks, both flexibility and the appropriate specifics for an application can be achieved.

In Section 2 we briefly describe the fundamental and classical properties considered by network science and those considered as being influenced by the meta-rules. In Section 3 we consider specific cases of meta-ruling for the linkage and in particular, linkage meta-ruled through constraints; linkage meta-ruled through the application of formal properties; meta-ruling configurations of a linkage, stating their general admissibility of occurrence; and superimposable meta-network(s) to be used in combination with the network under consideration. These approaches may apply to static, evolving, and dynamic networks and we conclude by stressing their conceptual and experimental nature.

2. NETWORK SCIENCE

The science of networks [1-4] represents systems as networks and systemic properties as network properties. For introductory purposes we mention only some features considered by network science, deriving from graph theory [5, pp. 287-302] and related, in statistical physics and mathematics, to the percolation theory that describes the behavior of a network when nodes or links are removed, a type of phase transition activated when the removal of a critical fraction of the network involves splitting into smaller connected clusters. There is a critical parameter at which the behavior of the system drastically changes, a phase transition occurs [6, 7].

Moreover, the linkage of networks is considered to be suitably influenced by the meta-rules, in order to achieve more flexibility and applicative adequacy in both design and when using real networks, e.g., in modeling and simulations but also in electronic systems.

In this section some properties of static (nodes never crash and links maintain invariable operational status) networks are described:

1) The *cluster coefficient* measures the structure of network nodes that are close to each other, or network cohesiveness. In networks it is possible to find clusters being subsets of the network and that possess a high degree of inner connectivity. The clustering coefficient measures the degree of clustering in a node's neighborhood. The cluster coefficient is considered as a measure of the likelihood that any two nodes having a common neighbor are themselves connected.

   In particular the local clustering coefficient is the ratio of the number of existing links connecting to each other the neighbors of the considered node to the maximum a priori, theoretical possible number of such links.

   Considering that the maximum possible number of links between N nodes is $\frac{N(N-1)}{2}$ the local clustering coefficient of the node i, denoted by Ci, is given by the formula:

   $$C_i = \frac{2e_i}{k_i(k_i-1)}$$

   where
   - k_i is the number of neighbors of the given node;
   - e_i is the number of connections between these neighbors.

   The global clustering coefficient of the whole network, denoted by C is then given by the average of the local clustering coefficients of the single nodes.

2) The degree of a node is the number of nodal neighbors, where the *degree distribution* is the probability of the

degree of a node across the entire network.

Taking in count undirected graphs, we can consider a square matrix $A=(a_{ij}),(i,j=1,...,N)$ whose generic element $a_{ij}=0$ if $i = j$ and $a_{ij}=1$ if there is a link connecting the node *i* with the node *j*, and in case this link is not present. This matrix is termed adjacency matrix. However, this definition presupposes that a node cannot have multiple links with another node nor a link with itself. Moreover, if the two nodes *i* and *j* are said *adjacent* or *neighbors*. Undirected graphs are characterized by the *degree of a node*, defined as the number of links connected with the node itself. The degree of the node *i* is denoted by k_i and is related to the adjacency matrix by the formula:

$$k_i = \sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji}$$

Furthermore, the *average degree* of a network $\langle k \rangle$ is the average value of node degree for all nodes of the network. Moreover, the characteristics of undirected graphs can be easily generalized to the directed graphs.

At this point we can consider the concept of *degree distribution P(k)* defined as the probability that a node of the network, randomly chosen from a uniform distribution, has the degree *k*.

3) The *fitness* is related by the way the links between nodes as they change over time, depending on the ability of nodes to attract links, for instance, suitable topological positions allow for shorter paths or enable accessibility to a greater variety of links, expressible as probability of attachment of a new node.

4) The *scale-freeness* occurs when the network has a high number of nodes with only a few links or a small number of nodes (termed hubs) with a high number of links. In scale-free networks, the probability that a node selected at random will possess a particular number of links, follows a power law [8]. Networks having power-law distributions are known as scale-free, because power laws have the same functional form at all scales. The property of a network being scale-free and scale-invariant [9] strongly correlates with its robustness by establishing fault tolerant behaviors [10, 11].

   In particular the *scale-free degree distribution* consists in the fact that that in some networks the statistical distribution of the node degrees has a dependence from the degrees themselves, as represented by a power law such as:

$$P(k) \approx Ak^{-\gamma}$$

   where the value of the exponent γ in many cases is $2 < \gamma < 3$.

5) The *small-world property*, occurring when most nodes are not close neighbors, but most nodes can be reached from every other node via a small number of intermediate links [12, 13]. Networks have a small value of average shortest path length while, at the same time, they have a high value for the global clustering coefficient. This property is also considered to increase robustness [14].

With regard to *type of networks*, these cases are of importance:

a) *evolving networks*, where the number of nodes and links is not fixed but changes with time as a function of a growth process, following the *Barabási-Albert model* [15]. The building process starts from an initial number of nodes and links and follows an iterative rule, which adds a new node at every step. In such networks the *clustering coefficient scales with network size*. This circumstance shows that this linkage differs from the small-world one, where the clustering coefficient is constant and independent of network size.

   The adding of new nodes is supposed to occur according to a criterion so-called “preferential attachment”. It consists on introducing, for each already existing node *i*, a probability *Pi* of attachment of the new node and it is given by a law like:

$$p_i = \frac{k_i}{\sum_j k_j}$$

   where k_i is the degree of the node *i* and the sum regards all previously existing nodes. This law promotes the nodes with higher number of links, receiving more links, while nodes with few links tend to be neglected. When the number of steps of this algorithm tends to infinity the degree distribution tends to a scale-free form given by:

$$P(k) \approx k^{-3}$$

   The clustering coefficient *C* scales with network size according to a law of the form:

$$C \approx N^{-0.75}$$

   This circumstance shows that this linkage differs from the small-world one, where t*he clustering coefficient is rather constant and independent from the network size*.

b) *dynamic networks*, where the dynamics relates to the fact that the network topology changes over time; nodes and/or links may come and go; links may crash and recover [16, 17].

With reference to the data transfer modes in networking, there are two possibilities:

- *synchronous networks* where data is transmitted and received at the same time. This is the case of full-duplex transmission. In such networks data is simultaneously transmitted in both directions and synchronization is accomplished with a signal clock.
- *asynchronous networks* do not use a signal clock to transmit or receive data, and data flows in only one direction at a time. This is also known as half-duplex and data transmission must occur intermittently, one at a time.

We do not consider here the case of quantum networks which requires a dedicated context [18].

3. META-RULES NETWORKS

As an extension to the previous conceptual cases, in this work the focus is on the concept of meta-ruled, static, evolving, or dynamic networks. The computational or operational execution of networks is implemented by applying and combining rules, where such rules concern the network function and particular linkage of nodes. The linkage presupposes, for instance, intra-link eligibility conditions, where such conditions may be context-dependent, for instance depending on topological properties of the interconnected nodes and on the permissible topological distance. Furthermore, in speculation, the possibility to prescribe a superimposed meta-network to take over on-demand, in combination with or as a partial localized replacement of the original network in certain contextual conditions.

However, such network meta-rules and the superimposed meta-network(s) are supposed to be suitable for computational and electronic networks, where a higher computational level of supervision is possible, such as in modeling and simulation. Examples include cellular automata, coupled maps, network models, simulations, and electronic circuits such as in [19, 20]. This approach may be noticeable in network model, dynamic and complex systems [21]; design and meta-rule of static, evolving, and dynamic networks are required to have some operational properties.

Such options are intended to have a soft, but crucial influence on the operation and properties of these networks in conceptual correspondence with the role of weak forces crucial in breaking equivalences, such as equilibria, starting collapses, and the initial conditions, e.g., for deterministic chaos and emergence [22].

Furthermore, the approaches outlined are in some way an extension to the use of multi layered and weighted links for Artificial Neural Networks [23] and variations of the artificial recurrent neural networks used for deep learning, to process not only single data points but also entire sequences of data [24].

This approach may be inadequate for phenomenological networked processes where supervision levels of meta-ruling and validation are difficult or not possible, such as for the Internet, phone networks, citation networks, blood flow and vessels, protein folding and neural networks in the brain. Nevertheless, phenomenological network processes may be, at least partially, represented and simulated by suitable supposed meta-ruled networks.

Meta-ruled linkage properties (constraints)

Considered herein, the case where the validity and acceptability of the general linkage of the network require the respect of some formal properties. The meaning of the expression ‘general linkage’ relates to the fact that it applies to both homogeneous and heterogeneous networks. In homogeneous networks all the nodes are supposed to have the same function. In heterogeneous networks, two or more classes of nodes are supposed, categorized by function and utility. Furthermore, we consider undirected links and those of any weight. Such linkage constraints are assumed to have suitable ranges of validity, which affect areas of the network or even the entire network. However, when applied in variable time regimes as temporal ranges of validity, the affected areas of the network are identifiable in different ways. Such as their topology, the admitted number of links for involved nodes, and in the configuration of the node linkage. Additionally, all the meta-ruling cases considered below may have regular application [25], recurrence rules, and are context-sensitive, i.e., activated if certain conditions are met.

Cases of meta-ruled linkage constraints include:

- topological constraints such as the inadmissibility of particular surrounding nodes linkage (e.g., in number of links, in the balance between the number of input and output links).
- the inadmissibility may relate to occurrence of a node linkage under a minimum topological distance and overcoming limits of recurrence.

Meta-ruled linkage

Examples of meta-rules where accordance of the general linkage is considered as validated, is required for admissible application. Meta-rules are supposed to govern the linkage and have, as in the previous case, suitable ranges of validity.

Cases of meta-rules for the general linkage include:

- if a node has n links in input, only *n-k* *(k<n)* links can discharge simultaneously according to a fixed or context-sensitive rule (for example the *n-k* links considered are those that at time t_n carries the signal with the highest or lower intensity or with any variance);
- if a node has *n* links in input and *m* links, with *m>n*, in output then the number of active, discharging admissible links in output are in number *k* = *m-n* chosen according to a rule (for example *k* is the number of links with greater intensity, or the node to which they discharge has number of links in input > *k*);
- nodes at a certain topological distance must have the same number of links active in input and/or in output.

Meta-ruling configurations of linkage

When considering examples of meta-ruling configuration, linkages whose accordance by the general linkage which it validates, is required for admissible application. Admissible meta-ruling configurations of a linkage are supposed to have, as in the previous case, suitable ranges of validity. Admissible configurations should occur, for instance, at suitable topological distances, no more than a suitable number of times, and according to limited variations. Cases of meta-ruling configurations for the general linkage include:

- non-admissibility or request of recurrence for specific sub-networks at suitable topological distances.
- minimum-maximal topological distance between sub-networks of the same type.
- properties of local micro sub-networks stating incompatibility or conditions of validity, for their linkage.
- incompatibility of the simultaneous active state for certain links input to the same node. Prioritization criteria are established.

At this point it is worth mentioning how the occurring of cases of inadmissibility and incompatibilities implies the non-operability of the implied sub-network(s) to be dealt with suitable replacement approaches related, for instance, to cases of single or multiple crashes of nodes or links, such as implementing alternative paths and spanning tree in a dynamic network.

Superimposing meta-network(s)

Meta-ruling, occurring through the simultaneous local or global replacement or suitable combinations of the network with another meta-network having, global or local, identical structure [21] are activated if certain conditions are met. The network is considered ‘meta’ since external, non-equivalent, introduced by an algorithmic choice, such as machine learning or operator decision, reminiscent of the role of the *oracle* in the machines of Turing [26]. It is also worth mention, the meta-network may be the same network at a different time, such as in the case of autocorrelation [27]. Where a network at a given time correlates with another at a different time, i.e., the correlation of the network with itself at a different point in time. This allows to reconstruct or anticipate the state of the network over time.

4. CONCLUSIONS

The conceptual possibilities introduced aim to add flexibility and suitability of network design and representation for systems. Such options are intended to have a soft, but crucial influence on the operation and properties of these networks. This conceptual correspondence is crucial in describing the role of weak forces capable of breaking equivalences, such as equilibria, starting collapses, and the initial conditions, e.g., for deterministic chaos and emergence. The approaches outlined are considerable as an extension to the use of multi layered and weighted links for Artificial Neural Networks and variations of the artificial recurrent neural networks used for deep learning, to process entire sequences of data.

The following research activities should be theoretically finalized to find generic properties of meta-ruled networks, such as those related to contextual adaptability as a form of artificial learning. Where autocorrelation over time allow for predictive and oriented behavioral and evolutionary features. The usage of a coupled meta-network(s) having possible layers and, even partially self-generated by the original network depending on the input process and self-regulatory-like computational processes. Subsequent research activities should also experimentally implement meta-ruled versions of current networks, typically though simulation and electronic circuits allowing suitable intervention to increase suitability to applications.

Conflict of Interest
The author declares no conflict of interest.

5. REFERENCES

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