New method for the failure probability of strict circular consecutive–k–out–of–n:F system

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Abstract

A recursive formula is given for calculating the failure probability for circular consecutive–k–out–of–n:F systems such that isolated strings of failures of length less than k (which do not cause system failure) do not occur, or are immediately corrected; ie, when system failure occurs it is because all failures present are in strings of length at least k. New method gives the failure probability function of circular strict system with time complexity $O(nk^2)$.

Keywords: Strict circular consecutive–k–out– of–n:F system; System failure probability; Recursive method.

1. Introduction

Previous methods of calculating the failure probability for consecutive–k–out–of–n:F systems, both recursive methods [1] and direct methods [4], do not rule out the situation that the failure mode (at least one string of k or more consecutive failures) is also accompanied by any possible strings of isolated failures of length less than k [2]. For example, in a strict consecutive–2–out–of–7:F system, F designates a failed component, and G designates a operational component; then the state $FGFGGFF$ is one of system failure only because of the last two F's, but there are also two isolated failure strings of length less than two. It seems reasonable to suppose, however, that in at least some applications of these systems, as might be the case with communication relay systems, isolated failure strings of length less than k –which may degrade performance but do not cause system failure–are, or can be, detected and corrected within an interval short enough that the normal operating mode can be considered to have no failed components. That is, Bollinger assumes that although prevention of loss of system continuity is important enough that a consecutive–k–out–of–n:F design is used for protection, the detection and repair or replacement of isolated failed components occur quickly enough that the context is not that of the ordinary consecutive–k–out–of–n:F system [2]. In such a case, system failure will occur when and only when k or more consecutive components fail, *and* without any isolated failure strings of fewer than k consecutive components. He called such system *strict* consecutive–k–out–of–n:F systems [2].

For example, let $n=7$, $k=2$, then the

following constitute failures of the linear system:

FFGGGGG
FFGGGFF
GGFFFGG
GFFFFFG
FFFGFFF .

While the following configurations would cause failure of the linear system, they are assumed *not to occur* because they contain isolated failure strings of length less than 2:

FGFFFFF
FFGFFGF
FFFGGFG .

The following represents a system in working order:

GGGGGGG .

As might be anticipated, the failure probability for an ordinary system is extremely conservative compared to that for a strict system, and when the strict system applies, it might be possible to use the information this provides in design economies. Bollinger studies only linear system with equal component probabilities [2].

Papastavridis studies linear system with un-equal component probabilities [3]. He provides a recursive formula for computing the system failure probability. Furthermore, he gave a simple exact formula for the failure probability of the system with components of equal probability.

Higashiyama distinguishes linear and circular forms of strict consecutive–k–out–of–n:F systems that are analogous to consecutive–k–out–of–n:F systems [5]. He estimated the time complexity of Papastavidis's recursive formula for computing the linear system failure probability and introduced a method to generate a computer enumeration for the failure probability function of the strict circular consecutive–k–out–of–n:F system. His method computes the failure probability in $O(nk^2)$ time, for the general case of un-equal component probabilities.

This paper presents new method to give another failure probability function of circular strict system with time complexity $O(nk^2)$.

2. Notation & Assumptions

Notation

- n number of components in the system.
- k minimum number of consecutive components which cause system failure, $2 \le k \le n$
- p_i operational probability of ith component.
- q_i $\equiv 1.0 - p_i$, failure probability of ith component.
- $F_L(i; j)$ failure probability (un-reliability) of the linear system with components $i, i+1, \ldots, j$.
- $F_C(n)$ failure probability (un-reliability) of the circular system with n components.

Assumptions

A. Each component and system is either operational or failed; the probabilities of component failure are known.
B. Component failures are mutually statistically independent.
C. Strings of failed components of length less than k do not occur, for example, are immediately repaired.
D. The system fails if and only if at least k consecutive components fail.

3. Linear system

Papastavridis studied the failure probability of strict linear consecutive–k–out–of–n:F system where all the components have un-equal failure probabilities . He has presented an equation to evaluate the failure probability (un-reliability) of the strict consecutive–k–out– of–n:F system as follows [2].

Let $F_L(1; n : G_n)$ and $F_L(1; n : F_n)$ represent the failure probability of the system with component n being operational or failure, respectively. Hence

$$F_L(1;n) = F_L(1;n:G_n) + F_L(1;n:F_n) \quad \text{for all } n > k. \qquad (1)$$

The system fails with the last component (component n) being operational if and only if the system fails at $(n-1)$ stages; hence

$$F_L(1;n:G_n) = p_n F_L(1;n-1) \qquad (2)$$

Denote $F_L^*(1;i) = \prod_{j=1}^{i} p_j + F_L(1;i)$,

for all $i \ge 1$, and $F_L^*(0;0) = 1.0$.

The same argument results in:

$$F_L(1;n:F_n) = q_n F_L(1;n-1:F_{n-1}) + p_{n-k} \cdot \prod_{i=n-k+1}^{n} q_i F_L^*(1;n-k-1) \qquad (3)$$

The recursive equations (1)–(3) give:

$$\begin{aligned} F_L(1;n) &= F_L(1;n:G_n) + F_L(1;n:F_n) \\ &= p_n F_L(1;n-1) + q_n F_L(1;n-1:F_{n-1}) \\ &\quad + p_{n-k} \prod_{i=n-k+1}^{n} q_i F_L^*(1;n-k-1) \\ &= F_L(1;n-1) - q_n\{(F_L(1;n-1) \\ &\qquad - F_L(1;n-1;F_{n-1})\} \\ &\quad + p_{n-k} \prod_{i=n-k+1}^{n} q_i F_L^*(1;n-k-1) \\ &= F_L(1;n-1) - p_{n-1} q_n F_L(1;n-2) \\ &\quad + p_{n-k} \prod_{i=n-k+1}^{n} q_i F_L^*(1;n-k-1) \qquad (4) \end{aligned}$$

For fixed $k \ge 2$ and for $n \ge k+1$, the recursive formula is:

$$F_L(1;n) = F_L(1;n-1) - p_{n-1} q_n F_L(1;n-2) + p_{n-k} \cdot \prod_{i=n-k+1}^{n} q_i \cdot F_L(1;n-k-1) + \prod_{i=1}^{n-k} p_i \cdot \prod_{j=n-k+1}^{n} q_j. \qquad (5)$$

The initial conditions are:

$F_L(1:i) = 0.0$, for $i = 0,1,2,\ldots,k-1$

and $F_L(1;k) = q_1 q_2 \ldots q_k$.

In the equation (5), first compute $Q(i) = \prod_{j=i-k+1}^{i} q_j$, for $i = 2k, 2k+1, \ldots, n$.

This requires $O(n+k) = O(n)$ time by first computing $O(k)$ and then computing $Q(i+1) = Q(i) q_{i+1} / q_{i-k+1}$ for each $i = 2k, 2k+1, \ldots, n-1$. Once this is done, then each $F_L(1;n)$ can be computed in constant time. Since there are n such $F_L(1;n)$'s to compute, we need another $O(n)$ time. Therefore the total time required is $O(n)$.

4. Circular system

Consider that the n components lie on a cycle. Suppose that the n components are labeled by the set $\{1,2,\ldots,n\}$ in a clockwise rotation (component n followed by component 1).

The component n has two states, operational or failed. So we have 2 events that the system is failed as follows:

A. If the component n is operational, the failure probability function is given by:

$$F_{CA}(n) = p_n \cdot F_L(1;n-1) \qquad (6)$$

B. If the k consecutive components containing of the component n are failed, these components cause the system failure.

For $n \ge k+1$, such a pair (s,l) must exist for the system to fail. Then the failure probability function is given by:

$$F_{CB}(n) = \sum_{s+n-l+1=k} \prod_{i=1}^{s} q_i \cdot \prod_{j=l}^{n} q_j$$

$$\cdot\{F_L^{*1}(s+1;l-1)+\prod_{t=s+1}^{l-1} p_t\} \quad (7)$$

$$F_L^{*1}(x;y)=q_x\cdot F_L^{*1}(x+1;y)$$
$$+p_x\cdot F_L^{*2}(x+1;y) \quad (8)$$

$$F_L^{*2}(x;y)=F_L^{*2}(x;y-1)\cdot q_y$$
$$+F_L(x;y-1)\cdot p_y \quad (9)$$

The initial conditions are: $F_{CB}(k)=q_1q_2\ldots q_k$

and $F_{CB}(i)=0.0$, for $i=0,1,\ldots,k-1$.

Note that $(s-1+n-l)=k$ is the number of failing components between component s and component l (clockwise from s to l).

Furthermore, using $m\,(0\le m\le k-1)$ equation (7) is rewritten as follows:

$$F_{CB}(n)=\sum_{m=0}^{k-1}\prod_{i=1}^{m}q_i\cdot\prod_{j=n-k+m+1}^{n}q_j$$

$$\cdot\{F_L^{*1}(m+1;n-k+m)+\prod_{t=m+1}^{n-k+m}p_t\} \quad (10)$$

The events, A and B, are disjoint each other. Then the system failure probability, $F_C(n)$, can be written as:

$$F_C(n)=F_{CA}(n)+F_{CB}(n) \quad (11)$$

Next consider the time complexity of equation (11). The equation (6) is computed in $O(n)$ time. In equation (10), there are at most k^2 distinct products:

$$\sum_{m=0}^{k-1}\prod_{i=1}^{m}q_i\cdot\prod_{j=n-k+m+1}^{n}q_j$$

which can be computed in $O(k^2)$. Therefore the total time required is $O(k^2)+O(nk^2)=O(nk^2)$.

5. Conclusion

This paper described a method to generate a computer enumeration for the failure probability function of the strict circular consecutive–k–out–of–n:F system. The method computes the failure probability in $O(nk^2)$ time, for the general case of un-equal component probabilities.

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